# Week 9: Assorted Problems

**1:** Find $\min_{a,b\in\mathbb{R}} \max(a^2+b, b^2+a)$.

**2:** Let $n$ be a positive integer and let $\zeta_n = e^{2\pi i/n}$. Let $a_1, \ldots, a_n \in \mathbb{R}$ be positive. Show that there is an $n$-gon with equal angles and side lengths $a_1, \ldots, a_n$ if and only if $a_1 + a_2\zeta_n + \cdots + a_n\zeta_n^{n-1} = 0$.

**3:** Find continuous functions $\phi, f, g, h : \mathbb{R} \to \mathbb{R}$ such that $\phi(x+y+z) = f(x) + g(y) + h(z)$ for all $x, y, z \in \mathbb{R}$.

**4:** Show that the product of the $2^{2019}$ numbers of the form $\pm 1 \pm \sqrt{2} \pm \cdots \pm \sqrt{2019}$ is the square of an integer.

**5:** Let $n$ be a positive integer and let $z$ be a root of $x^n - x - 1$. Show $2\mathrm{Re}\left(z - \dfrac{1}{z}\right) > \dfrac{1}{|z|^2} - 1$. (This can be used to show $x^n - x - 1$ is irreducible.)

**6:** Show that a calculator capable of only doing the binary operation $(x, y) \mapsto 1/(x-y)$ can be used as a proper calculator for addition and multiplication.

**7:** Let $(a_n)$ be a sequence of positive integers. Show that there exists a real number $\gamma > 1$ such that infinitely many $a_n$ appear as $\lfloor \gamma^k \rfloor$ for some positive integer $k$.

**8:** Let $f(x) = a_n x^n + \cdots + a_0$ be a polynomial with complex coefficients with roots $r_1, \ldots, r_n$. Define its Mahler measure by $M(f) = |a_n| \prod_{i=1}^{n} \max(1, |r_i|)$. Show that $M(f) \leq \sqrt{|a_0|^2 + |a_1|^2 + \cdots + |a_n|^2}$.

**9:** Suppose $A$ and $B$ are two subsets of $\mathbb{Z}/p\mathbb{Z}$ such that $|A|+|B| \leq p$. Let $A+B = \{a+b \colon a \in A, b \in B\}$. Show that $|A+B| \geq |A| + |B| - 1$.

**10:** Let $(a_n)$ be an increasing sequence of positive integers such that $|a_{n+1} - a_n| \leq 2019$ for all $n$. Show that there are infinitely many pairs $(i, j)$ with $i < j$ and $a_i \mid a_j$.

**11:** (a) (Capelli's theorem) Let $K$ be a subfield of $\mathbb{C}$. Suppose $f(x) \in K[x]$ is irreducible. Let $\alpha \in \mathbb{C}$ be a root of $f(x)$ and let $g(x) \in K[x]$ such that $g(x) - \alpha$ is irreducible in $K[\alpha][x]$. Show that $f(g(x))$ is irreducible in $K[x]$.

(b) Let $f(x) \in \mathbb{Z}[x]$ be monic. Let $p$ be a prime number. Suppose $f(x)$ is irreducible in $\mathbb{Z}[x]$ and $(-1)^{\deg f} f(0)$ is not a $p$-th power in $\mathbb{Q}$. Show that $f(x^p)$ is irreducible in $\mathbb{Z}[x]$.

(c) Let $n$ be a positive integer. Show that the polynomial $(x^2+1^2)(x^2+2^2)\cdots(x^2+n^2)+1$ is irreducible in $\mathbb{Z}[x]$.

Hints

**1:** Add them.

**2:** Place the $n$-gon on the complex plane with the first edge from 0 to $a_1$, the second edge from $a_1$ to $a_1 + a_2\zeta_n$, etc.

**3:** Show $\phi(x+y) = \phi(x) + \phi(y) - f(0) - g(0) - h(0)$.

**4:** If $P(x)$ is an even polynomial, then so is $P(x+\sqrt{k})P(x-\sqrt{k})$.

**5:** Write $z = re^{i\theta}$. Then $1 + 2r\cos\theta = |z+1|^2 - r^2 = r^{2n} - r^2$.

**6:** Show that $x - y$ and $1/x$ are enough to recover multiplication.

**7:** Find nested decreasing closed intervals $[\alpha_k, \beta_k]$ so that for every $x \in [\alpha_k, \beta_k]$, the floor of some power of $x$ belongs to the given sequence. Cantor's nested interval theorem implies that the intersection of these closed intervals is nonempty.

**8:** For any large $N$, let $z_1, \ldots, z_N$ be the $N$-th roots of unities. Then $\displaystyle\sum_{i=1}^{N} |f(z_j)|^2 = N \cdot \sum_{j=0}^{n} |a_j|^2$. Apply the AM-GM inequality to conclude $\displaystyle\left(\sum_{j=0}^{n} |a_j|^2\right)^{\frac{1}{2}} \geq |a_n| \prod_{i=1}^{n} \sqrt[N]{|1 - r_i^N|}$. Take limit as $N \to \infty$.

**9:** We want to replace $A$ and $B$ by $A \cap B$ and $A \cup B$ and then apply induction of $|A|$. Show first one can apply shifts so that $A \cap B$ is a proper subset of $A$.

**10:** Construct an infinite list of 2019 tuples of consecutive integers $(a_{n,1}, \ldots, a_{n,2019})$ such that $a_{i,k} \mid a_{j,k}$ whenever $i \leq j$ for all $k$. Apply the Pigeonhole principle to any set of 2020 lists.

**11:** (a) Let $\beta \in \mathbb{C}$ be a root of $g(x) - \alpha$. Then $[K[\alpha, \beta] : K] = \deg f \deg g$ but $K[\alpha, \beta] = K[\beta]$. (b) It suffices to show if $\alpha$ is a root of $f$, then $\alpha$ is not a $p$-th power in $\mathbb{Q}[\alpha]$. Suppose $\alpha = u(\alpha)^p$ for some $u(x) \in \mathbb{Q}[x]$. Then $f(x) \mid u(x)^p - x$. Show that this implies that $(-1)^{\deg f} f(x)$ is a $p$-th power. (c) It suffices to show $g(x) = (x+1^2)\cdots(x+n^2)+1$ is irreducible. Write $g(x) = F(x)G(x)$. Then for $j = -1^2, \ldots, -n^2$, $F(j) = G(j) = \pm 1$. So $F = G$. But $g(0)$ is not a square.