Week 8: Assorted Problems

1: Prove the identity $\sum_{k=1}^{n}(k^2+1)k! = n(n+1)!$.

2: Show that any integer $n > 6$ can be written as the sum of two coprime integers both at least 2.

3: Find all polynomials satisfying $(x+1)P(x) = (x-10)P(x+1)$.

4: Show that there is no infinite arithmetic progression whose terms are all perfect squares.

5: Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = f(x+\sqrt{2}) = f(x+\sqrt{3})$ for all $x \in \mathbb{R}$,

6: Let a and b be integers. Suppose for any positive integer n, $a^n + n$ divides $b^n + n$. Show that $a = b$.

7: Find the average value of $\sum_{k=1}^{n} |a_{2k} - a_{2k-1}|$ over all permutations $\{a_1, \ldots, a_{2n}\}$ of $\{1, \ldots, 2n\}$.

8: Find all functions $f : [0,\infty) \to [0,\infty)$ that are differentiable at $x = 1$ such that $f(x^3)+f(x^2)+f(x) = x^3 + x^2 + x$ for all x.

9: Show that for any positive integer k, the sum of digits of any multiple of $10^k - 1$ is at least $9k$.

10: Suppose (a_n) is a non-decreasing sequence of positive integers such that $\lim_{n\to\infty} \frac{a_n}{n} = 0$. Show that the sequence $\left(\frac{n}{a_n}\right)$ takes every positive integer values. (In particular, this implies for any positive integer k, there is an integer N such that there are exactly N primes less than kN.)

11: Let $a_1, \ldots, a_k$ be positive real numbers such that at least one of them is not an integer. Prove that there exist infinitely many positive integers n such that n and $\lfloor a_1 n\rfloor + \lfloor a_2 n\rfloor + \cdots + \lfloor a_k n\rfloor$ are coprime.

12: (a) Suppose $f(x)$ is a monic integer polynomial with nonzero constant term. Suppose $f(x)$ has exactly one (complex) root of absolute value greater than 1. Show that $f(x)$ is irreducible in $\mathbb{Z}[x]$.

(b) (Perron's Criterion) Suppose $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ with $a_0 \neq 0$ and $|a_{n-1}| > 1 + |a_0| + |a_1| + \cdots + |a_{n-2}|$. Show that $f(x)$ is irreducible in $\mathbb{Z}[x]$.

Hints

1: Induction.

2: Consider the four cases $n = 4m, 4m+1, 4m+2, 4m+3$.

3: Show $x \mid P(x)$ and $x - 10 \mid P(x)$.

4: If (a_n^2) is an arithmetic progression, then $a_{n+1} - a_n$ is decreasing.

5: For any $m, n \in \mathbb{Z}$, $f(0) = f(m\sqrt{2} + n\sqrt{3})$. The fractional part $\{k\sqrt{3/2}\}$ is dense.

6: Suppose p is a prime that doesn't divide $b - a$. Pick n such that $n \equiv 1 \pmod{p-1}$ and $n \equiv -a \pmod{p}$.

7: It suffices to find the average of $|a_2 - a_1|$ and multiply by n.

8: Let $h(t) = f(e^t) - e^t$. Show that for any $\epsilon > 0$, $|h(t)/t| < \epsilon$ for all t.

9: Let $s(N)$ denote the sum of the digits of N. Show that given any positive integer $N > 10^k$, there exists an integer $M < N$ congruent to N mod $10^k - 1$ with $s(M) \leq s(N)$.

10: Consider the finite set $\{k \colon \frac{a_{mk}}{mk} \geq \frac{1}{m}\}$

11: Suppose otherwise. Then for a sequence of prime numbers p_n going to infinity, $(\lfloor a_1 n \rfloor + \lfloor a_2 n \rfloor + \cdots + \lfloor a_k n \rfloor)/p_n$ is an integer sequence converging $a_1 + \cdots + a_k$.

12: (a) Take any factorization $f(x) = g(x)h(x)$ into monic integer polynomials. We may assume all the roots of g have absolute values less than or equal to 1. Show that they can't all have absolute value 1. (b) Show first that $f(x)$ has no roots of absolute value 1. Then let r be a root of absolute value greater than 1. Show that all the roots of $f(x)/(x - r)$ have absolute value less than 1.