Week 7: Assorted Problems

1: Solve the system of linear equations $x_1 + x_2 + x_3 = x_2 + x_3 + x_4 = \cdots = x_{99} + x_{100} + x_1 = x_{100} + x_1 + x_2 = 0$.

2: Show that for any positive integer n, $\sqrt[n]{n} < 1 + \sqrt{\frac{2}{n-1}}$.

3: Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(f(x))) = x$ for all $x \in \mathbb{R}$.

4: Show that $\left\lfloor \frac{p}{q} \right\rfloor + \left\lfloor \frac{2p}{q} \right\rfloor + \cdots + \left\lfloor \frac{(q-1)p}{q} \right\rfloor = \left\lfloor \frac{q}{p} \right\rfloor + \left\lfloor \frac{2q}{p} \right\rfloor + \cdots + \left\lfloor \frac{(p-1)q}{p} \right\rfloor$ where p and q be coprime positive integers.

5: Let ϕ denote the Euler-phi function. Show that for any positive integers a and n, $n \mid \phi(a^n - 1)$.

6: Show that any prime divisor of the n-th Fermat number $2^{2^n} + 1$ is congruent to 1 modulo 2^{n+1}.

7: Let $f(x) = \sum_{k=1}^{n} a_k \sin(kx)$ with $a_1, \ldots, a_n \in \mathbb{R}$ and $n \geq 1$. Suppose $|f(x)| \leq |\sin x|$ for all x. Show that $\left| \sum_{k=1}^{n} k a_k \right| \leq 1$.

8: Solve in positive integers the equation $x^{x+y} = y^{y-x}$.

9: Fix any positive $r > 0$. Draw a ball of radius r at every nonzero lattice point ($\mathbb{Z}^3 - \{(0,0,0)\}$) in $\mathbb{R}^3$. Show that any line that passes through the origin will intersect some ball.

10: Let $n \geq 3$. Show that $\sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \sum_{i=1}^{n} |i - \sigma(i)| = 0$.

11: Let f be a polynomial with integer coefficient with $f(0) = 0$, $f(1) = 1$. Suppose p is a prime such that $f(k) \equiv 0, 1 \pmod{p}$ for all integers k. Show that $\deg f \geq p - 1$.

12: Let $p \equiv 3 \pmod 4$ be a prime. Show that $\prod_{1 \leq x < y \leq \frac{p-1}{2}} (x^2 + y^2) \equiv (-1)^{\lfloor \frac{p+1}{8} \rfloor} \pmod{p}$.

Hints

1: Add all the equations together.

2: Let $x = \sqrt[n]{n} - 1$. Expand $(1+x)^n$ and consider the x^2 term.

3: Show f is bijective and then strictly increasing.

4: Consider the box $[0,p] \times [0,q]$ and the line joining $(0,0)$ and (p,q). How many lattice points are there in the box (including the boundary) that lie beneath the line?

5: What is the order of a in $\mathbb{Z}_{a^n-1}^{\times}$?

6: Let k be the order of 2 mod p. Then $k \mid 2^{n+1}$. If $k \mid 2^n$, then $2^k - 1 \mid 2^{2^n} - 1$.

7: Compute $f'(0)$ by definition.

8: Use the prime factorizations of x and y to show $x \mid y$.

9: Take a cylinder symmetric around the origin, centered around the line of radius r long enough to have volume more than 8. By Minkowski's Theorem, it contains a nonzero lattice point.

10: Consider the matrix $A(x) = (x^{|i-j|})_{i,j}$. Then the desired sum equals $\left.\frac{d}{dx}\det(A(x))\right|_{x=1}$.

11: Suppose $\deg(f) \le p-2$ and apply the interpolation formula with $f(0), \ldots, f(p-1)$. Show that the coefficient of x^{p-1} is not 0 mod p.

12: Take a primitive root a mod p and set $x = a^2$. Then the product on the left is $\prod(x^i + x^j)$. Compare the two Vandermonde determinant $\prod(x^i - x^j)$ and $\prod(x^{2i} - x^{2j})$.