Week 6: Assorted Problems

1: Let $f_0(x) = 0$ and $f_n(x) = \sqrt{x + f_{n-1}(x)}$ for $n \geq 1$. Compute the limit $\lim_{x\to\infty} \big(f_n(x) - f_{n-1}(x)\big)$.

2: Prove the "$T2$ Lemma": for real numbers $a_1, \ldots, a_n$ and positive real numbers $x_1, \ldots, x_n$,

$$\frac{a_1^2}{x_1} + \cdots + \frac{a_n^2}{x_n} \geq \frac{(a_1 + \cdots + a_n)^2}{x_1 + \cdots + x_n}.$$

3: Let n be any positive integer and suppose a is an integer coprime to n. Show that there exist integers $0 < x, y \leq \sqrt{n}$ such that $ax \equiv \pm y \pmod{n}$.

4: Let S denote the set of positive integers congruent to 1 mod 3. Let T denote the set of positive integers congruent to 1 mod 4. Does there exist a bijection $f : S \to T$ such that $f(mn) = f(m)f(n)$ for all $m, n \in S$?

5: Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function and $a \in (0, 1)$ such that $\lim_{x\to 0} f(x) = 0$ and $\lim_{x\to 0} \dfrac{f(x) - f(ax)}{x} = 0$. Show that $\lim_{x\to 0} \dfrac{f(x)}{x} = 0$.

6: Given distinct integers $x_1, x_2, \ldots, x_n$, prove that $\prod_{i<j}(x_i - x_j)$ is divisible by $1!2!\cdots(n-1)!$

7: Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f(x) = f(x^2)$ for all $x \in \mathbb{R}$. Show that f is constant.

8: Suppose $f(x)$ and $g(x)$ are two integer polynomials with no common complex roots. For every integer n, let $a_n = \gcd(f(n), g(n))$. Show that the sequence $\{a_n\}$ is periodic.

9: Show that for any positive $a_1, \ldots, a_5 \in \mathbb{R}$, $\dfrac{a_1}{a_2 + a_3} + \dfrac{a_2}{a_3 + a_4} + \dfrac{a_3}{a_4 + a_5} + \dfrac{a_4}{a_5 + a_1} + \dfrac{a_5}{a_1 + a_2} \geq \dfrac{5}{2}$.

10: Show that for any integer $n > 1$, $n \nmid 2^{n-1} + 1$.

11: A continuous function $f : [0, 1) \to [0, \infty)$ satisfies $f(\frac{1}{2}x + \frac{1}{2}) = f(x) + 1$ and $f(1 - x) = \dfrac{1}{f(x)}$. Compute $\displaystyle\int_0^1 f(x)\, dx$.

12: Suppose a and b are positive integers such that for every prime p, $a \pmod{p} \leq b \pmod{p}$. Show that $a = b$.

Hints

1: $f_n(x) - f_{n-1}(x) = \dfrac{f_{n-1}(x) - f_{n-2}(x)}{f_n(x) + f_{n-1}(x)} \leq \dfrac{1}{2\sqrt{x}}(f_{n-1}(x) - f_{n-2}(x)).$

2: Cauchy-Schwartz.

3: Apply Pigeonhole to $ax - y \pmod{n}$ for $0 \leq x, y \leq \sqrt{n}$.

4: There are infinitely many primes $\equiv 1 \pmod 4$, $\equiv 3 \pmod 4$, $\equiv 1 \pmod 3$, $\equiv 2 \pmod 3$.

5: Estimate $|f(x) - f(a^n x)|$ for x in a good range and let n go to infinity.

6: Consider the determinant of the $n \times n$ integer matrix whose (i, j)-entry is $\binom{x_i}{j-1}$. Relate that to the Vandermonde polynomial.

7: The sequence $x_1 = x$ and $x_n = \sqrt{x_{n-1}}$ converges to 1.

8: The condition on f and g implies that there exist integer polynomials F, G and a positive integer A such that $fF + gG = A$. Show that $a_n = a_{n+A}$ for all n.

9: Apply the T2 Lemma to $\dfrac{a_1^2}{a_1(a_2+a_3)} + \dfrac{a_2^2}{a_2(a_3+a_4)} + \dfrac{a_3^2}{a_3(a_4+a_5)} + \dfrac{a_4^2}{a_4(a_5+a_1)} + \dfrac{a_5^2}{a_5(a_1+a_2)}$.

10: n is a product of odd primes. Take a prime divisor p of n such that $p-1$ is divisible by the smallest power of 2. Work mod p.

11: Show first that $f(x) + f(\frac{1}{2} - x) = 1$. Then let $d_n = \displaystyle\int_{1-2^{-n}}^{1-2^{1-n}} f(x)dx$. Find recursive formulas for d_n.

12: Show first that $\dbinom{b}{a} \mid \displaystyle\prod_{p \leq a} p^{m(p)-1}$ where $m(p)$ is the largest k such that p^k divides a factor in the product $b(b-1)\cdots(b-a+1)$. Then reduce to the case $b \geq 2a$ and deduce that $\dbinom{b}{a}$ is too big in this case.