Week 5: Assorted Problems

1: Suppose a is a rational number such that $a^2 - a$ is an integer. Show that a is an integer.

2: Find the smallest n such that there exists sets A and B with $|A| = |B| = n$ and $A+B = \{a+b\colon a \in A, b \in B\} = \{0, 1, 2, \ldots, 2019\}$.

3: Let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of x. Show that $\lim_{n\to\infty} \{(2+\sqrt{3})^n\} = 1$

4: Show that for any integer $n > 1$, $n \nmid 2^n - 1$.

5: Suppose $f : \mathbb{N} \to \mathbb{N}$ such that $\sum_{d|n} f(d) = n^2$. Let $\phi(n)$ denote the Euler-phi function. Show that $\frac{f(n)}{\phi(n)} = n \prod_{p|n} \big(1 + \frac{1}{p}\big)$ where the product is taken over all prime divisors of n.

6: Find all polynomials $P(x, y)$ of two variables such that $P(a, b)P(c, d) = P(ac + bd, ad + bc)$ for all a, b, c, d.

7: Suppose $P(x)$ is a polynomial such that for any integer n, the equation $P(x) = n$ has a rational solution. Show that all the coefficients of $P(x)$ are rational. (In fact, $P(x) = ax + b$.)

8: Suppose $P(x)$ is a polynomial with real coefficients such that $P(x) \geq 0$ for all $x \geq 0$. Prove that there are two real polynomials $A(x)$ and $B(x)$ such that $P(x) = A(x)^2 + xB(x)^2$.

9: Find all solutions to the equation $9^x + 11^{x^2} = 10^x + 10^{x^2}$.

10: Suppose f is a function with continuous derivative on $[0, 1]$ such that $0 < f'(x) \leq 1$ for all x. Suppose $f(0) = 0$. Show that $\left(\int_0^1 f(x)\,dx\right)^2 \geq \int_0^1 f(x)^3\,dx$.

11: Let $a_1 = 1$ and $a_{n+1} = a_n + \lfloor \sqrt{a_n} \rfloor$ for $n \geq 1$. What is the smallest n such that $a_n > 2019$?

12: Let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of x. Order the solutions to $\sqrt{\lfloor x \rfloor \lfloor x^3 \rfloor} + \sqrt{\{x\}\{x^3\}} = x^2$ with $x \geq 1$ from smallest to largest by $x_1, x_2, x_3, \ldots$. Give a formula for $\sum_{k=1}^{n} \frac{1}{x_{2k}^2 - x_{2k-1}^2}$.

Hints

1: Apply the Rational Root Theorem to $x^2 - x - (a^2 - a)$.

2: $|A + B| \le |A||B|$. Guess the smallest n and write down such A and B.

3: $(2+\sqrt{3})^n + (2-\sqrt{3})^n \in \mathbb{Z}$.

4: Let p be the smallest prime divisor of n. Then $p \mid 2^d - 1$ where $d = \gcd(n, p-1)$.

5: Show first f is multiplicative. Then prove the result for prime powers.

6: Let $Q(x,y) = P(\frac{x+y}{2}, \frac{x-y}{2})$. Then $Q(ab, cd) = Q(a,c)Q(b,d)$.

7: Write $x_n \in \mathbb{Q}$ such that $P(x_n) = n$. Suppose $P(x)$ has degree d. The equations $P(x_k) = k$ for $k = 0, 1, \ldots, d$ determine the coefficients of $P(x)$. For the extended claim, the denominators of x_k are bounded above and so by the Mean value theorem, there is a sequence approaching ∞ for which the derivative of P is bounded. A polynomial with bounded derivative is linear.

8: $(A^2 + xN^2)(C^2 + xD^2) = (AC - xBD)^2 + x(AD + BC)^2$.

9: Let $f(t) = t^{x^2} - (20-t)^x$. Then $f(10) = f(11)$. Apply Rolle's Theorem.

10: Let $F(t) = \left(\int_0^t f(x)\,dx\right)^2 - \int_0^t f(x)^3\,dx$. Enough to show $F'(t) \ge 0$ for all t. Use the same idea for the next step.

11: The perfect squares that show up in the sequence are precisely all the powers of 4. Then $\lfloor a_n \rfloor$ takes every positive integer value twice, except for all powers of 2 where they appear three times each.

12: Cauchy-Schwartz gives $ac + bd \le \sqrt{a^2+b^2}\sqrt{c^2+d^2}$. Then write $\{x\}$ in terms of $\lfloor x \rfloor$.