Week 4: Assorted Problems

1: Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(f(x))) = x^3$ and $f(f(f(f(f(x))))) = x^5$ for every $x \in \mathbb{R}$.

2: Consider the Fibonacci sequence defined by $f_1 = 1$, $f_2 = 1$, $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$. Find a general formula for $\sum_{k=1}^{n} f_k 2^{-k}$.

3: Show that for any positive integer n, the sequence $2, 2^2, 2^{2^2}, 2^{2^{2^2}}, \ldots \pmod n$ is eventually constant.

4: Suppose $P(x)$ is a cubic polynomial such that $(x^3 - 2x + 1 - P(x))(2x^3 - 5x^2 + 4 - P(x)) \leq 0$ for all $x \in \mathbb{R}$ and $P(0) = 3$. Find $P(-1)$.

5: Suppose A, B and C are three 2-digit perfect squares such that the 6-digit number $10000A+100B+C$ is also a perfect square. Find all possible triples (A, B, C).

6: Suppose $c \in \mathbb{R}$ such that one of the roots of the equation $x^3 - \frac{3}{4}x + c = 0$ belongs to $[-1, 1]$. Show that all the roots belong to $[-1, 1]$.

7: Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $|f(x + y) - f(x - y) - y| \leq y^2$ for all $x, y \in \mathbb{R}$.

8: For a positive integer n, compute the integral $\displaystyle\int \frac{x^n}{1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}}\, dx$.

9: Let $x_1, \ldots, x_k$ be real numbers such that the set $\{\cos(n\pi x_1) + \cos(n\pi x_2) + \cdots + \cos(n\pi x_k) \colon n \geq 1\}$ is finite. Show that $x_i \in \mathbb{Q}$ for all i.

10: Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a twice-differentiable function such that $xf''(x) + f'(x) + f(x) \leq 0$ for all $x > 0$. Find $\lim\limits_{x\to\infty} f(x)$.

11: Let $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ be two distinct sets of positive integers such that any integer can be written as $a_i + a_j$ with $i \neq j$ in exactly as many ways as it can be written as $b_i + b_j$ with $i \neq j$. Show that n is a power of 2.

12: Let $a_0 = 11$, $b_0 = 1$ and $a_{n+1} = a_n + a_{n-1}b_n^{2019}$, $b_{n+1} = b_n b_{n-1}$ for $n \geq 1$. Find the smallest positive integer k such that $11 \mid a_k$ and $11 \mid b_k - 1$ for any such sequence where a_1, b_1 are positive integers and $11 \nmid b_1$.

Hints

1: $f(f(f(f(f(x))))) = f(f(x^3))$.

2: Compute the first few terms and observe a pattern.

3: If $\gcd(a, n) = 1$ and $r \equiv s \pmod{\phi(n)}$, then $a^r \equiv a^s \pmod{n}$.

4: Let $Q(x) = x^3 - 2x + 1$ and $R(x) = 2x^3 - 5x^2 + 4$. Show $P(x) = tQ(x) + (1 - t)R(x)$ for some $t \in \mathbb{R}$.

5: Show the only possibility is $10000A + 100B + C = (100A + C)^2$.

6: $4\cos^3\theta - 3\cos\theta = \cos(3\theta)$.

7: Set $g(x) = f(x) - x/2$. Show that $g'(x) = 0$ for all x.

8: Take $f(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$. Then the integrand is $n!(f - f')/f$.

9: Denote the n-th term by a_n. Then the set of k-tuples $(a_n, a_{2n}, \ldots, a_{kn})$ is also finite. Apply Pigeonhole to find $n < m$ such that $\cos(n\pi x_i) = \cos(m\pi x_{\sigma(i)})$ for some permutation σ of $\{1, \ldots, k\}$.

10: Consider $g(x) = x(f'(x)/f(x)) + x$.

11: Let $f(x) = \sum x^{a_i}$ and $g(x) = \sum x^{b_i}$. Then $f(x)^2 - f(x^2) = g(x)^2 - g(x^2)$.

12: Work modulo 11. Then $b_n^{2019} \equiv b_n^9 \pmod{11}$. Raising to the 9-th power is a bijection modulo 11, so we may replace b_n^9 by b_n. Set $b_1 = c$ and then $b_n = c^{f_n}$ where f_n is the n-th Fibonacci number. One may also assume $a_1 = 1$. Then $a_n = \sum_{k=0}^{f_n - 1} c^k$.