Week 3: Assorted Problems

1: Let $a_1, a_2, \ldots$ be a sequence of real numbers such that for all n, $\sum_{k=1}^{n} a_k \left(\frac{k}{n}\right)^2 = 1$. Find a closed formula for a_n.

2: Suppose $f(x) = x^3 + ax^2 + bx + c$ is a polynomial with three distinct negative integer roots. Suppose $a + b + c = 1104$. Find c.

3: Let a sequence be defined as follows: $a_0 = 1$ and for $n > 0$, $a_n = \frac{1}{3}a_{n-1}$ with probability $\frac{1}{2}$ and $a_n = \frac{1}{9}a_{n-1}$ with probability $\frac{1}{2}$. What is the expected value of $\sum_{n=1}^{\infty} a_n$.

4: Let $a_0 = 1$, $a_1 = 1$, $a_2 = 2$, and for $n \geq 3$, define a_n to be the last digit of the sum of the proceeding three terms in the sequence. Determine whether or not the string 1001 occurs somewhere in this sequence.

5: Prove that the limit $\lim_{n\to\infty} \frac{1}{n^2} \prod_{i=1}^{n} (n^2 + i^2)^{1/n}$ exists and find its value.

6: Let $x_1, \ldots, x_{2n}$ be real numbers such that when any one of them is removed, the rest can be partitioned into two sets of equal sum. Show that all the x_i's are 0.

7: Let $f : [a, b] \to (a, b)$ be a continuous function. Show that for every natural number n, there exists a positive number α and $c \in (a, b)$ such that $f(c) + f(c + \alpha) + \cdots + f(c + n\alpha) = (n + 1)(c + \frac{n}{2}\alpha)$.

8: Compute the integral $\int_0^2 \frac{\ln(1 + x)}{x^2 - x + 1}\, dx$.

9: Let $a_1 > 0$ and $a_{n+1} = a_n + \frac{n}{a_n}$ for $n \geq 1$. Show that $\lim_{n\to\infty} n(a_n - n)$ exists.

10: Let $A, B \in M_n(\mathbb{C})$ be $n \times n$ matrices such that $AB - BA$ commutes with A. Show that $AB - BA$ is nilpotent; that is, there exists a positive integer k such that $(AB - BA)^k = 0$.

11: Show that there is no function $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) > 0$ and $f(x + y) \geq f(x) + yf(f(x))$ for all $x, y \in \mathbb{R}$.

12: 1) Show that for any string of numbers, there exists infinitely many powers of 2 that start with that string.

2) Prove that there exists an integer N such that for any $n \geq N$, there are more powers of 2 within $2^1, 2^2, \ldots, 2^n$ that start with 7 than 8.

Hints

1: Work out the first few terms.

2: $(x_1-1)(x_2-1)(x_3-1) = -a-b-c-1$.

3: Find $E[a_n]$ for each n.

4: The sequence is periodic.

5: Spread the $1/n^2$ in each product, take log, and recognize it as a Riemann sum.

6: $(x_1, \ldots, x_{2n})$ is the kernel of a $2n \times 2n$ matrix with 0 on the diagonal and ± 1 elsewhere.

7: Set $h(x) = f(x) - x$. Then h is positive in a neighborhood around b and negative in a neighborhood around a.

8: Take $u = x+1$, then $v = 3/u$.

9: Show $a_n - n$ is positive and decreasing.

10: Let $C = AB - BA$. Show that $\mathrm{tr}(C^m) = 0$ for all m.

11: Show first that $f(f(z))$ is positive for some z. Then show that f is eventually increasing to infinity. Find some a so that $f(a) \geq a+1$.

12: For any irrational number α, the fractional part of $n\alpha$ as n varies is dense. For the second part, $0.8 \times 0.875 = 0.7$, $0.9 \times 0.876 = 0.7884$.