Week 2: Assorted Problems

1: Let $x = 0.12345678910111213\ldots$ be the number whose decimal expansion consists of the sequence of natural numbers written next to each other. Show that x is irrational.

2: Define the sequence $(x_n)_{n=1}^{\infty}$ by $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2}^{x_n}$ for $n \geq 1$. Prove that the sequence x_n converges and find its limit.

3: Let a, b, c, d be positive integers such that $ab = cd$. Show that $a^2 + b^2 + c^2 + d^2$ is composite.

4: Determine all the values of the positive integer $n \geq 2$ such that $(x_1 + \cdots + x_n)^2 \geq n(x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1)$ for all nonnegative real numbers $x_1, \ldots, x_n$.

5: Let $n \geq 2$ be a positive integer. Show that every complex number c with $|c| \leq n$ can be written as $c = a_1 + a_2 + \cdots + a_n$ where $|a_j| = 1$ for every j.

6: Does there exist a function $g : \mathbb{Q}^2 \to \mathbb{Q}$ such that $g(p, y) \in \mathbb{Q}[y]$ whenever $p \in \mathbb{Q}$, and $g(x, q) \in \mathbb{Q}[x]$ whenever $q \in \mathbb{Q}$, but g is not a polynomial?

7: Let a_n be the fractional part of $\ln(n)$ for $n \geq 1$. Let b_n be the average of $a_1, \ldots, a_n$. Find a continuous function f on $[0, 1]$ such that $\lim_{n\to\infty} \big(b_n - f(a_n)\big) = 0$.

8: Let $f(x)$ be a monic integer polynomial all of whose (complex) roots have absolute values at most 1. Show that all the roots of $f(x)$ are roots-of-unities. *This is Kronecker's Theorem.*

9: Let n be a positive integer with at least 4 positive divisors. Let the four smallest positive divisors be $a = 1 < b < c < d$. Find all possible n such that $n^2 = 1 + b^3 + d^3$.

10: Let $f : [0, 1] \times [0, 1] \to \mathbb{R}$ be a continuous function. Find the limit

$$\lim_{n\to\infty} \left(\frac{(2n+1)!}{(n!)^2}\right)^2 \int_0^1 \int_0^1 \big(xy(1-x)(1-y)\big)^n f(x, y)\, dx dy.$$

11: Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a function with $f(x + f(y + xy)) = (y + 1)f(x + 1) - 1$ for all $x, y \in \mathbb{R}^+$. Show that $f(x) = x$.

12: For any polynomial $f(x)$ with all real roots, let $d(f)$ denote the distance between its largest real root and its smallest real root. For any positive integer n, find the largest real number $C(n)$ such that for every polynomial $f(x)$ of degree n with all real roots, $d(f') \geq C(n)d(f)$.

Hints

1: The decimal expansion of a rational number is eventually periodic.

2: Show the function $f(x) = \sqrt{2}^x - x$ is increasing in the desired range.

3: Take out some common factors to express each of a, b, c, d as a product.

4: Pick some values for the x_i's to give a bound for n.

5: Construct a continuous function that is a sum of n functions that always have absolute value 1 so that it takes the values 0 and n.

6: Yes, use the countability of $\mathbb{Q}$ to write down such a function using infinite sums.

7: Take $e^k \leq n < e^{k+1}$ and estimate a_n and b_n.

8: Let $a_1, \ldots, a_n$ be the roots of $f(x)$. Take $f_r(x) = \prod(x - a_i^r)$ for any positive integer r. The coefficients of f_r are absolutely bounded. Apply the Pigeonhole principle to f_{2^i} as i varies over positive integers.

9: Observe that b is a prime and d is not divisible by b.

10: Consider first the case when $f(x) = x^k y^l$ is a polynomial.

11: First prove f is injective. Then choose y to get rid of the -1 on the right-hand-side.

12: Show first that by moving two consecutive roots closer, one obtains a polynomial g with $d(g') \leq d(f')$.