Week 1: Assorted Problems

1: Let $n_1, \ldots, n_k$ be k integers and let $m_1, \ldots, m_k$ be a permutation of them. Show that $|n_1 - m_1| + |n_2 - m_2| + \cdots + |n_k - m_k|$ is even.

2: Let u and v be positive real numbers. Minimize the larger of $2u + v^{-2}$ and $2v + u^{-2}$.

3: Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f(x) \leq x$ and $f(x+y) \leq f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Show that $f(x) = x$ for all $x \in \mathbb{R}$.

4: Let $(a_n)_{n=1}^{\infty}$ be a sequence of nonnegative real numbers such that $1 + a_{m+n} \leq (1 + a_m)(1 + a_n)$ for all $m, n \in \mathbb{N}$. Show that the sequence $(x_n)_{n=1}^{\infty}$ defined by $x_n = \sqrt[n]{1 + a_n}$ converges.

5: Let $S(n) = \displaystyle\sum_{m=1}^{n} \frac{1}{\langle \sqrt{m} \rangle}$, where $\langle x \rangle$ denotes the integer closest to x. Give a general formula for $S(n^2)$.

6: Give an example of positive integers a, b, c, d, e such that a, b^2, c^3, d^4, e^5 is a non-constant arithmetic progression.

7: Suppose that $P_1, P_2, \ldots, P_6$ are points in $\mathbb{R}^3$. Let D be the 6×6 matrix whose (i,j)-entry is the square of the distance between P_i and P_j. Show that $\det(D) = 0$.

8: Let n be a positive integer. Suppose that $f(x)$ is differentiable on $[0,1]$ with $f(0) = 0$ and $f(1) = 1$. Prove that there exist n (distinct) numbers $x_1, \ldots, x_n$ in $(0,1)$ for which $\displaystyle\sum_{i=1}^{n} \frac{1}{f'(x_i)} = n$.

9: Suppose $f(x)$ is twice-differentiable on $[0,1]$ and $f(0)f(1) \geq 0$. Prove

$$\int_0^1 |f'(x)|\,dx \leq 2\int_0^1 |f(x)|\,dx + \int_0^1 |f''(x)|\,dx.$$

10: Let $S_n = \left(\frac{1}{n}\right)^n + \left(\frac{2}{n}\right)^n + \cdots + \left(\frac{n-1}{n}\right)^n$. Compute $\displaystyle\lim_{n\to\infty} S_n$.

11: Let $\mathbb{Q}[x]$ denote the vector space over $\mathbb{Q}$ of polynomials with rational coefficients in x. Find all $\mathbb{Q}$-linear maps $\Phi : \mathbb{Q}[x] \to \mathbb{Q}[x]$ that send irreducible polynomials to irreducible polynomials.

12: Which integers can be written in the form $\dfrac{(x+y+z)^2}{xyz}$ where x, y, z are positive integers?

Hints

1: How are the parities of x and $|x|$ related?

2: The standard trick to bounding $u + u^{-2}$ is to apply the AMGM inequality to $u/2 + u/2 + u^{-2}$.

3: $x = 0 + x$, $0 = x + (-x)$.

4: First show x_n is bounded below and above and then show $\lim x_n = \inf x_n$.

5: Find the contribution of $1/\langle\sqrt{m}\rangle$ from all such m where $\langle\sqrt{m}\rangle = k$.

6: Scaling an arithmetic progression gives another arithmetic progression.

7: Write D as the sum of 5 rank 1 matrices.

8: Pick $n - 1$ convenient values between 0 and 1 to apply the intermediate value theorem and then the mean value theorem.

9: Suppose $|f'(x)|$ takes a minimum of m at $x = x_0$ and a maximum of M at $x = x_1$.

10: For a fixed k, $(1 - k/n)^n$ approaches e^{-k} from below.

11: If f and g are two polynomials such that $f + cg$ is irreducible for all $c \in \mathbb{Q}$, then either $g = 0$ or f is degree 1 and g is a constant.

12: Show one can assume $x \leq y \leq z \leq x+y$ and use it to find a small upper bound for $(x+y+z)^2/xyz$.