Solutions to the Bernoulli Trials Problems for 2019

1: For every function $f : \mathbf{N} \to \mathbf{N}$ with $0 \le f(n) \le n$ for all $n \in \mathbf{N}$, the graph of f contains an infinite set of colinear points.

Solution: This is FALSE. For example we can take $f(x) = \lfloor \sqrt{x} \rfloor$.

2: For $n = \prod_{i=1}^{l} {p_i}^{k_i}$ where $l \in \mathbf{Z}^+$, each $k_i \in \mathbf{Z}^+$ and the p_i are distinct primes, let $f(n) = \sum_{i=1}^{l} k_i p_i$. Then $\sum_{n=2}^{\infty} \frac{1}{f(n)}$ converges.

Solution: This is FALSE. When $n = 2^k$ we have $f(n) = 2k$ so $\sum_{n=1}^{\infty} \frac{1}{f(n)} \ge \sum_{k=1}^{\infty} \frac{1}{f(2^k)} = \sum_{k=1}^{\infty} \frac{1}{2k} = \infty$.

3: For all integers $n \ge 3$, if $\varphi(n) = \varphi(n-1) + \varphi(n-2)$ then n is prime.

Solution: This is FALSE. For example, when $n = 1037 = 17 \cdot 61$ so that $\varphi(n) = 16 \cdot 60 = 960$, we have $n-1 = 2^2 \cdot 7 \cdot 37$ so that $\varphi(n-1) = 2 \cdot 6 \cdot 36 = 432$ and $n-2 = 1035 = 3^2 \cdot 5 \cdot 23$ so that $\varphi(n-2) = 6 \cdot 4 \cdot 22 = 528$.

4: For every integer $n \ge 2$ there exists a nonzero $n \times n$ matrix A with entries in $\mathbf{Z}$ such that if we interchange any two rows in the matrix A then the resulting matrix B is skew-symmetric, that is $B^T = -B$.

Solution: This is FALSE. Let $n \ge 2$ and suppose that A is such a matrix. When $k \ne l$ we must have $A_{k,l} = 0$, otherwise interchanging rows k and l would give a matrix B with $B_{l,l} = A_{k,l} \ne 0$ but then $B^T \ne -B$. This shows that A must be a diagonal matrix. Also, if $k \ne l$, we must have $A_{k,k} = -A_{l,l}$ because when we interchange rows k and l we obtain a matrix B with $B_{k,l} = A_{l,l}$ and $B_{l,k} = A_{k,k}$. Thus A must be diagonal with $A_{k,k} = -A_{l,l}$ for all $k \ne l$. This can only occur when $n = 2$.

5: There exists a sequence $\{a_n\}_{n \ge 1}$ where each $a_n \in \mathbf{R}^2$ with $a_n \to 0$ such that the open discs $D\big(a_n, \frac{1}{n}\big)$ are disjoint.

Solution: This is TRUE. For each $n \in \mathbf{Z}^+$, place discs of radius $\frac{1}{n^2}, \frac{1}{n^2+1}, \cdots, \frac{1}{(n+1)^2-1}$ in a vertical column with the bottom disc, of radius $\frac{1}{n^2}$ sitting in a $\frac{2}{n^2} \times \frac{2}{n^2}$ square with its bottom left vertex at position $a_{n^2} = \big(\frac{\pi^2}{3} - \sum_{k=1}^{n} \frac{2}{k^2}\, , 0\big)$. Note that $a_{n^2} \to (0,0)$ as $n \to \infty$ and that the height of the column of discs above a_{n^2} is equal to $2\big(\frac{1}{n^2} + \frac{1}{n^2+1} + \cdots + \frac{1}{(n+1)^2-1}\big) \le 2 \cdot \frac{1}{n^2} \cdot ((n+1)^2 - n^2) = \frac{2(2n+1)}{n^2} \to 0$ as $n \to \infty$.

6: The closed unit square in $\mathbf{R}^2$ is equal to the union of a collection of disjoint sets each of which is homeomorphic to the open interval $(0,1)$.

Solution: This is TRUE. For example one set, which is homeomorphic to $(0,1)$, can follow the boundary of the square starting and ending at the lower left corner $(0,0)$, and another can follow the V shape with vertices at $\big(\frac{1}{2}, 1\big)$, $(0,0)$ and $(1,1)$, and the remaining portion of the square is the disjoint union of three open triangles which can be covered by disjoint horizontal open line segments.

7: There is a unique positive integer n such that there exists a connected planar graph G with n vertices each of which has degree 5.

Solution: This is FALSE. The icosahedron is one such graph. Given two such graphs G_1 and G_2 we can form a third such graph, with more vertices, as follows. Choose external edges $e_1 = (u_1, v_1)$ on G_1 and $e_2 = (u_2, v_2)$ on G_2, then delete the edges e_1 and e_2 and replace them by the edges (u_1, v_1) and (u_2, v_2).

8: For all $n, l \in \mathbf{Z}^+$, there exists a map $f : \mathbf{Z}_{n^l} \to \mathbf{Z}_n$ such that every sequence of length l in $\mathbf{Z}_n$ is of the form $f(k+1), f(k+2), \cdots, f(k+l)$ for some $k \in \mathbf{Z}_{n^l}$.

Solution: This is TRUE. We can construct such a map f as follows. Let G be the directed graph whose vertices are the n^{l-1} sequences of length $l-1$ in $\mathbf{Z}_n$ with an edge from the vertex $(a_1, a_2, \cdots, a_{l-1})$ to each of the n vertices $(a_2, a_3, \cdots, a_{l-1}, x)$ with $x \in \mathbf{Z}_n$. Note that G has n^l edges which correspond to the n^l sequences $(a_1, a_2, \cdots, a_{l-1}, x)$ of length l. Note that G is connected, indeed a walk from the vertex $(a_1, \cdots, a_{l-1})$ to the vertex $(b_1, \cdots, b_{l-1})$ is given by

$$(a_1, a_2, \cdots, a_{l-1}),\ (a_2, a_3, \cdots, a_{l-1}, b_1),\ (a_3, \cdots, a_{l-1}, b_1, b_2), \cdots, (a_{l-1}, b_1, \cdots, b_{l-2}),\ (b_1, b_2, \cdots, b_{l-1}).$$

Also note that each vertex has the same number of incoming edges as it has outgoing edges, indeed the vertex $(a_1, a_2, \cdots, a_{l-1})$ has the n incoming edges $(x, a_1, \cdots, a_{l-2})$ with $x \in \mathbf{Z}_n$. Since G is connected and each vertex has the same number of incoming and outgoing edges, it follows that G admits an Eulerian cycle (that is a directed cycle which traverses every edge). Given an Eulerian cycle

$$(a_1, \cdots, a_{l-1}),\ (a_2, \cdots, a_l),\ (a_3, \cdots, a_{l+1}), \cdots,\ (a_{n^l}, a_1, \cdots, a_{l-2}),\ (a_1, \cdots, a_{l-1})$$

we obtain a function $f : \mathbf{Z}_{n^l} \to \mathbf{Z}_n$ as desired by setting $\big(f(1), f(2), \cdots, f(n^l)\big) = (a_1, a_2, \cdots, a_{n^l})$.

Here is a sketch of a proof that such a graph G admits an Eulerian cycle. First choose a directed cycle C in G of maximal length from the vertex $(00\cdots0)$ to itself. Suppose, for a contradiction, that this cycle does not traverse every edge of G. The collection of all the missing edges (along with their endpoints) forms a nonempty subgraph H of G. Since each vertex of G has the same number of incoming and outgoing edges, and since H is obtained from G by removing the edges in the cycle C, it follows that each vertex of H has the same number of incoming and outgoing edges. Since G is connected, it follows that one of the vertices in H lies along the cycle C. Starting at a vertex v which lies in both C and H, we form a path in H by adding one edge at a time until we obtain a maximal path D in H. Note that this maximal path D must be a cycle ending at the initial vertex v because H has the property that each vertex in H has the same number of incoming and outgoing edges. This gives us a contradiction because the cycle D in H can be inserted into the cycle C to obtain a cycle in G of greater length than C.

9: There exists an uncountable set S of subsets of $\mathbf{Z}$ with the property that for all $A, B \in S$ with $A \neq B$ the set $A \cap B$ is finite.

Solution: This is TRUE. For example, we can construct such a set S as follows. Let T be the (uncountable) set of all binary sequences $\alpha = (a_0, a_1, a_2, \cdots)$ with $a_0 = 1$. To each $\alpha = (a_0, a_1, a_2, \cdots) \in T$, we associate the set $A(\alpha) \subseteq \mathbf{Z}$ whose elements are the numbers whose binary representations are $a_0, a_0a_1, a_0a_1a_2, \cdots$. Then we can take $S = \big\{A(\alpha)\big|\alpha \in T\big\}$.

10: There exists a sequence of sets $A_1, A_2, A_3, \cdots$ where each A_n is an n-element set of positive real numbers with $\prod\limits_{a \in A_n} a = 1$ such that $\lim\limits_{n\to\infty} \left(\frac{1}{n} \sum\limits_{a \in A_n} a\right) = 1$.

Solution: This is TRUE. For example, we can choose

$$A_1 = \{1\}\ ,\ A_{2m} = \bigcup_{k=1}^{m} \left\{ \tfrac{2^k+1}{2^k}\ ,\ \tfrac{2^k}{2^k+1} \right\} \ \text{ and } \ A_{2m+1} = A_{2m} \cup \{1\}.$$

It is clear that each A_n has n elements and the product of the elements in A_n is equal to 1. Let $S_n = \sum\limits_{a \in A_n} a$ and note that

$$S_{2m} = \sum_{k=1}^{m} \left(\tfrac{2^k+1}{2^k} + \tfrac{2^k}{2^k+1}\right) = \sum_{k=1}^{m} \left(1 + \tfrac{1}{2^k} + 1 - \tfrac{1}{2^k+1}\right) = 2m + \sum_{k=1}^{m} \tfrac{1}{2^k} - \sum_{k=1}^{n} \tfrac{1}{2^k+1}.$$

Thus we have $S_{2m} < 2m + \sum\limits_{k=1}^{m} \frac{1}{2^k} < 2m+1$ and we have $S_{2m} > 2m - \sum\limits_{k=1}^{m} \frac{1}{2^k+1} > 2m - \sum\limits_{k=1}^{m} \frac{1}{2^k} > 2m-1$ so that $\frac{2m-1}{2m} < \frac{1}{2m} S_{2m} < \frac{2m+1}{2m}$. By the Squeeze Theorem, it follows that $\lim\limits_{m\to\infty} \frac{1}{2m} S_{2m} = 1$. Since $S_{2m+1} = S_{2m} + 1$ we have $2m < S_{2m+1} < 2m+2$ so that $\frac{2m}{2m+1} < \frac{1}{2m+1} S_{2m+1} < \frac{2m+2}{2m+1}$ and hence $\lim\limits_{m\to\infty} \dfrac{1}{2m+1} S_{2m+1} = 1$.

11: For every sequence $\{a_n\}_{n\geq 1}$ in $\mathbf{R}$, if $\lim\limits_{n\to\infty}\dfrac{1}{n}\displaystyle\sum_{k=1}^{n}a_k = b\in\mathbf{R}$ and $\lim\limits_{n\to\infty}\dfrac{1}{\log n}\displaystyle\sum_{k=1}^{n}\dfrac{a_k}{k} = c\in\mathbf{R}$ then $b=c$.

Solution: This is TRUE. Let $S_n=\sum\limits_{k=1}^{n}a_k$ and suppose that $\lim\limits_{n\to\infty}\frac{S_n}{n}=b$. Let $\epsilon>0$ and choose ℓ so that $\left|\frac{S_k}{k}-b\right|<\epsilon$ for all $k\geq\ell$. By Abel's Summation by Parts Formula, for $n>\ell$ we have

$$\sum_{k=1}^{n}\tfrac{a_k}{k}=\tfrac{1}{n}S_n+\sum_{k=1}^{n}S_k\left(\tfrac{1}{k}-\tfrac{1}{k+1}\right)=\tfrac{1}{n}S_n+\sum_{k=1}^{n-1}\tfrac{S_k}{k(k+1)}=\tfrac{1}{n}S_n+\sum_{k=1}^{\ell}\tfrac{S_k/k}{k+1}+\sum_{k=\ell+1}^{n-1}\tfrac{S_k/k}{k+1}$$

and hence $c=\lim\limits_{n\to\infty}\frac{1}{\log n}\sum\limits_{k=1}^{n}\frac{a_k}{k}=\lim\limits_{n\to\infty}\frac{1}{\log n}\sum\limits_{k=\ell+1}^{n-1}\frac{S_k/k}{k+1}$. Also, we have

$$(b-\epsilon)\ln\tfrac{n+1}{\ell+2}\leq(b-\epsilon)\sum_{k=\ell+1}^{n-1}\tfrac{1}{k+1}\leq\sum_{k=\ell+1}^{n-1}\tfrac{S_k/k}{k+1}\leq(b+\epsilon)\sum_{k=\ell+1}^{n-1}\tfrac{1}{k+1}\leq(b+\epsilon)\ln\tfrac{n}{\ell+1}$$

and hence (by dividing by $\log n$ and taking the limits) $(b-\epsilon)\leq c\leq(b+\epsilon)$. Since $\epsilon>0$ was arbitrary, $c=b$.

12: $\displaystyle\int_0^\infty \ln^2\left(\tfrac{x}{x+3}\right)dx\geq 10.$

Solution: This is FALSE. In fact, $\displaystyle\int_0^\infty \ln^2\left(\tfrac{x}{x+3}\right)dx=\pi^2<10$. Let $t=\ln\left(\frac{x+3}{x}\right)$ so that $e^t=\frac{x+3}{x}=1+\frac{3}{x}$ hence $x=\frac{3}{e^t-1}$. Then, using Integration by Parts (twice), we have

$$\begin{aligned}\int_0^\infty \ln^2\left(\tfrac{x}{x+3}\right)dx&=\int_0^\infty \ln^2\left(\tfrac{x+3}{x}\right)dx=\int_\infty^0 t^2\,d\left(\tfrac{3}{e^t-1}\right)=\left[\frac{3t^2}{e^t-1}\right]_\infty^0-\int_\infty^0\frac{6t}{e^t-1}\,dt\\&=\int_0^\infty\frac{6t}{e^t-1}\,dt=\int_0^\infty\frac{6t\,e^{-t}}{1-e^{-t}}\,dt=\int_0^\infty 6t\left(e^{-t}+e^{-2t}+e^{-3t}+\cdots\right)dt\\&=\sum_{k=1}^\infty\int_0^\infty 6t\,e^{-kt}\,dt=\sum_{k=1}^\infty\left(\left[-\tfrac{6}{n}\,te^{-kt}\right]_0^\infty+\int_0^\infty\tfrac{6}{n}\,e^{-kt}\,dt\right)\\&=\sum_{k=1}^\infty\left[-\tfrac{6}{n^2}\,e^{-kt}\right]_0^\infty=\sum_{k=1}^\infty\tfrac{6}{n^2}=\pi^2.\end{aligned}$$

13: $1+6\cos\frac{2\pi}{7}\geq 2\sqrt{7}\,\cos\left(\frac{1}{3}\arctan 3\sqrt{3}\right)$.

Solution: This is TRUE, indeed equality holds. We have

$$\begin{aligned}\cos(7\theta)=\mathrm{Re}\Big(\big(\cos\theta+\mathrm{i}\sin\theta\big)^7\Big)&=\cos^7\theta-21\cos^5\theta\sin^2\theta+35\cos^3\theta\sin^4\theta-7\cos\theta\sin^6\theta\\&=\cos^7\theta-21\cos^5\theta(1-\cos^2\theta)+35\cos^3\theta(1-\cos^2\theta)^2-7\cos\theta(1-\cos^2\theta)^3\\&=64\cos^7\theta-112\cos^5\theta+56\cos^3\theta-7\cos\theta.\end{aligned}$$

and so $\cos\frac{2\pi}{7}$ is a root of the polynomial

$$\begin{aligned}f(x)&=64x^5-112x^5+56x^3-7x-1\\&=(x-1)(64x^6+64x^5-48x^4-48x^3+8x^2+8x+1)\\&=(x-1)(8x^3+4x^2-4x-1)^2\end{aligned}$$

and hence a root of $g(x)=8x^3+4x^2-4x-1$. Indeed, $\cos\frac{2\pi}{7}$ is the unique positive root of $g(x)$ and the other two roots are $\cos\frac{4\pi}{7}$ and $\cos\frac{6\pi}{7}$. It follows that $6\cos\frac{2\pi}{7}$ is the positive root of $f\left(\frac{x}{6}\right)=\frac{1}{27}x^3+\frac{1}{9}x^2-\frac{2}{3}x-1$ or, equivalently, of $g(x)=27f\left(\frac{x}{6}\right)=x^3+3x^2-18x-27$. Thus $1+6\cos\frac{2\pi}{7}$ is a root of

$$h(x)=g(x-1)=(x-1)^3+3(x-1)^2-18(x-1)-27=x^3-21x-7.$$

Since $h(0)=-7$, $h(-1)=13$ and $\lim\limits_{x\to-\infty}h(x)=-\infty$, the other 2 roots of $h(x)$ are negative, so $1+6\cos\frac{2\pi}{7}$ is the unique positive root of $h(x)$. To solve $h(x)=0$ we let $x=w+\frac{7}{w}$ to get

$$\begin{aligned}x^3-21x-7=0&\iff\left(w-\tfrac{7}{w}\right)^3-21\left(w-\tfrac{7}{w}\right)-7=0\iff w^3-\tfrac{7^3}{w^3}-7=0\iff w^6-7w^3-7^3=0\\&\iff w^3=\tfrac{7\pm\sqrt{7^2-4\cdot7^3}}{2}=\tfrac{7}{2}\left(1\pm3\sqrt{3}\,i\right)=7\sqrt{7}e^{\pm i\theta}\iff w=\sqrt{7}e^{\pm i(\theta+2\pi k)/3},\ k\in\{0,1,2\}\end{aligned}$$

where $\theta=\tan^{-1}3\sqrt{3}$. When $w=\sqrt{7}e^{\pm i\theta}$ we have $x=w+\frac{7}{w}=\sqrt{7}\left(e^{i\theta/3}+e^{-i\theta/3}\right)=2\sqrt{7}\cos\left(\frac{\theta}{3}\right)$.

14: There exists a continuous function $f : [0,1] \to \mathbf{R}$ which crosses the x-axis at uncountably many points, where we say that f crosses the x-axis at a when $f(a) = 0$ and for all $\delta > 0$ there exist $x, y \in (a-\delta, a+\delta)$ with $f(x) < 0$ and $f(y) > 0$.

Solution: This is TRUE. Let C be the (standard) Cantor set and let $I_{n,k}$ be the open intervals removed when constructing C as follows. We let $C_0 = [0,1]$ and we let $I_{0,1} = \left(\frac{1}{3}, \frac{2}{3}\right)$. Let $C_1 = C_0 \setminus I_{0,1} = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$ and let $I_{1,1} = \left(\frac{1}{9}, \frac{2}{9}\right)$ and $I_{1,2} = \left(\frac{7}{9}, \frac{8}{9}\right)$. At the n^{th} step, C_n is the disjoint union of 2^n closed intervals each of size $\frac{1}{3^n}$ and $I_{n,k}$, for $1 \le k \le 2^n$, are the open middle thirds of the closed component intervals of C_n. The Cantor set C is the intersection $C = \bigcap_{n=1}^{\infty} C_n$. It is well-known that C is uncountable and, indeed, that there is an injective map F from the set of binary sequences into C given by $F(a_1, a_2, \cdots) = \sum_{k=1}^{\infty} \frac{2a_k}{3^k}$. For each n, k, let $g_{n,k} : [0,1] \to [0, 2^{-n}]$ be a continuous function with $g_{n,k}(x) = 0$ for $x \notin I_{n,k}$ and $0 < g_{n,k}(x) \le 2^{-n}$ for $x \in I_{n,k}$, and then let $f = \sum_{n,k} (-1)^n g_{n,k}$. Then f is continuous (since the sum converges uniformly by the Weierstrass M-test) and f crosses the x-axis at every point in C.

15: For $n \in \mathbf{Z}^+$ and $x \in \mathbf{R}$, define $f_n : \mathbf{R} \to [0,1)$ by $f_n(x) = nx - \lfloor nx \rfloor$. Then for some $a < b$, the sequence of functions $\{f_n : [a,b] \to \mathbf{R}\}$ has a convergent subsequence.

Solution: This is FALSE. We claim that for every bounded integrable function f of period 1, which is not equivalent to a constant, it is impossible for a subsequence of $\{f(nx)\}$ to converge on any nondegenerate interval $[a,b]$. For any interval $[\alpha, \beta]$, we have

$$
\begin{aligned}
\int_\alpha^\beta f(nx)\,dx &= \frac{1}{n} \int_{n\alpha}^{n\beta} f(x)\,dx \\
&= \frac{1}{n} \left(\sum_{k=\lfloor n\alpha \rfloor}^{\lfloor n\beta \rfloor - 1} \int_k^{k+1} f(x)\,dx + \int_{\lfloor n\beta \rfloor}^{n\beta} f(x)\,dx - \int_{\lfloor n\alpha \rfloor}^{n\alpha} f(x)\,dx \right) \\
&= \frac{1}{n} \big(\lfloor n\beta \rfloor - \lfloor n\alpha \rfloor \big) \int_0^1 f(x)\,dx + o(1) \\
&= (\beta - \alpha) \int_0^1 f(x)\,dx + o(1) \quad \text{as } n \to \infty.
\end{aligned}
$$

Thus

$$\lim_{n\to\infty} \int_\alpha^\beta f(nx)\,dx = (\beta - \alpha) K \quad (1)$$

where $K = \int_0^1 f(x)\,dx$. Now consider a subsequence $\{f(n_k x)\}$ of $\{f(nx)\}$ and suppose that $f(n_k x) \to g(x)$ for all $x \in [a,b]$. By the Dominated Convergence Theorem, if $[\alpha, \beta] \subseteq [a,b]$ we have

$$\int_\alpha^\beta g(x)\,dx = \lim_{k\to\infty} \int_\alpha^\beta f(n_k x)\,dx = (\beta - \alpha) K.$$

Thus

$$\int_\alpha^\beta \big(g(x) - K \big)\,dx = 0.$$

Because α and β are arbitrary in $[a,b]$. we must have $g(x) = K$ almost everywhere in $[a,b]$. Apply equation (1) to the function $|f(x) - K|$ and use the Dominated Convergence Theorem to get

$$(b-a) \int_0^1 \big| f(x) - K \big|\,dx = \lim_{k\to\infty} \int_a^b \big| f(n_k x) - K \big|\,dx = \int_a^b |g(x) - K|\,dx = 0.$$

Thus $f(x) = K$ almost everywhere in $[a,b]$