Solutions to the Bernoulli Trials Problems for 2015

1: The number of positive integers whose digits occur in strictly decreasing order is $2(2^9-1)$.

Solution: This is TRUE. Every such positive integer is obtained from one of the 2^n subsets of the set $\{9,8,7,\cdots,1,0\}$ (after arranging the digits in the subset in decreasing order). We do not include the empty set or the set $\{0\}$, so there are 2^n-2 such numbers.

2: Let n be the smallest positive integer such that $7^n \equiv 1 \bmod 2015$. Then $n \geq 100$.

Solution: This is FALSE. Note that $2015 = 5\cdot 13\cdot 31$. By Fermat's Little Theorem we have $7^4 = 1 \bmod 5$, $7^{12} = 1 \bmod 13$ and $7^{30} = 1 \bmod 31$ and so $7^{60} = 1$ modulo 5, 13 and 31. By the Chinese Remainder Theorem, $7^{60} = 1 \bmod 2015$.

3: The number $\sqrt[3]{7+5\sqrt{2}}+\sqrt{11-6\sqrt{2}}$ is rational.

Solution: This is TRUE. Note that $(1+\sqrt{2})^3 = 1+3\sqrt{2}+6+2\sqrt{2} = 7+5\sqrt{2}$ and $(3-\sqrt{2})^2 = 9-6\sqrt{2}+2 = 11-6\sqrt{2}$ and so $\sqrt[3]{7+5\sqrt{2}}+\sqrt{11-6\sqrt{2}} = (1+\sqrt{2})+(3-\sqrt{2}) = 4$.

4: For every field F and every square matrix A with entries in F, we have $\text{Row}(A)\cap\text{Null}(A) = \{0\}$.

Solution: This is FALSE. When $F = \mathbf{C}$ and $A = \begin{pmatrix} 1 & i \\ 1 & i \end{pmatrix}$ we have $\text{Row}(A) = \text{Null}(A) = \text{Span}\big\{(1,i)^T\big\}$. As another example, when $F = \mathbf{Z}_5$ and $A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ we have $\text{Row}(A) = \text{Null}(A) = \text{Span}\big\{(1,2)^T\big\}$.

5: For each $n \in \mathbf{Z}^+$, let x_n be the number of matrices $A \in M_{3\times n}(\mathbf{Z}_3)$ with no two horizontally or vertically adjacent entries equal. Then there exists $n \in \mathbf{Z}^+$ such that x_n is a square.

Solution: This is FALSE. Let a_n be the number of such $3\times n$ matrices which end with a column of the form (x,y,x) (with x and y distinct) and let b_n be the number of such $3\times n$ matrices which end with a column of the form (x,y,z) (with x, y and z distinct). Note that $a_1 = 6$ and $b_1 = 6$ and $x_n = a_n + b_n$ for all $n \geq 1$. Note that the column $(0,1,0)$ can be followed by any of the columns $(1,0,1)$, $(1,0,2)$, $(1,2,1)$, $(2,0,1)$ and $(2,0,2)$ and the column $(0,1,2)$ can be followed by any of the columns $(1,0,1)$, $(1,2,0)$, $(1,2,1)$ and $(2,0,1)$, and so we have $a_{n+1} = 3a_n + 2b_n$ and $b_{n+1} = 2a_n + 2b_n$. The first few values of a_n, b_n and x_n are given by

n	1	2	3	4
$\frac{1}{6}a_n$	1	5	23	105
$\frac{1}{6}b_n$	1	4	18	82
$\frac{1}{6}x_n$	2	9	41	187

An easy induction argument shows that for $n \geq 2$, $\frac{1}{6}b_n$ is even and $\frac{1}{6}a_n$ and $\frac{1}{6}x_n$ are odd. Since x_n is equal to 6 times an odd number, it cannot be a square.

6: $\prod\limits_{k=1}^{50} \frac{2k}{2k-1} > 12$.

Solution: This is TRUE. Let $p = \prod\limits_{k=1}^{50} \frac{2k}{2k-1} > 12$. Then we have

$$\begin{aligned} p^2 &= \frac{2}{1}\,\frac{2}{1}\,\frac{4}{3}\,\frac{4}{3}\,\frac{6}{5}\,\frac{6}{5}\,\frac{8}{7}\,\frac{8}{7}\,\frac{10}{9}\,\frac{10}{9}\cdots\frac{100}{99}\,\frac{100}{99} > \frac{2}{1}\,\frac{2}{1}\,\frac{4}{3}\,\frac{4}{3}\,\frac{6}{5}\,\frac{6}{5}\cdot\frac{8}{7}\,\frac{9}{8}\,\frac{10}{9}\,\frac{11}{10}\cdots\frac{100}{99}\,\frac{101}{100} \\ &= \frac{2}{1}\,\frac{2}{1}\,\frac{4}{3}\,\frac{4}{3}\,\frac{6}{5}\,\frac{6}{5}\cdot\frac{101}{7} = \frac{2^8\cdot 101}{5^2\cdot 7} > \frac{2^8\cdot 100}{25\cdot 7} = \frac{1024}{7} > \frac{1022}{7} = 146 > (12)^2. \end{aligned}$$

7: $\displaystyle\int_0^{\pi/2} \sqrt{2\tan x}\; dx > \pi.$

Solution: This is FALSE. Make the substitution $u = \sqrt{2\tan x}$ so that $u^2 = 2\tan x$, hence

$$u\,du = \sec^2 x\; dx = (1+\tan^2 x)\,dx = \big(1+\tfrac{u^4}{4}\big)\,dx = \big(\tfrac{u^4+4}{4}\big)\,dx$$

to get

$$\begin{aligned}
\int_{x=0}^{\pi/2} \sqrt{2\tan x}\; dx &= \int_{u=0}^{\infty} \frac{4u^2}{u^4+4}\; du = \int_0^\infty \frac{4u^2\; du}{(u^2-2u+2)(u^2+2u+2)} \\
&= \int_0^\infty \frac{u}{u^2-2u+2} - \frac{u}{u^2+2u+2}\; du = \int_0^\infty \frac{(u-1)+1}{(u-1)^2+1} - \frac{(u+1)-1}{(u+1)^2+1}\; du \\
&= \Big[\tfrac{1}{2}\ln\big((u-1)^2+1\big) + \tan^{-1}(u-1) - \tfrac{1}{2}\ln\big((u+1)^2+1\big) + \tan^{-1}(u+1)\Big]_0^\infty \\
&= \Big[\tfrac{1}{2}\ln\Big(\frac{u^2-2u+1}{u^2+2u+1}\Big) + \tan^{-1}(u-1) + \tan^{-1}(u+1)\Big]_0^\infty \\
&= \big(0+\tfrac{\pi}{2}+\tfrac{\pi}{2}\big) - \big(0-\tfrac{\pi}{4}+\tfrac{\pi}{4}\big) = \pi.
\end{aligned}$$

8: A light at position $(0,0,4)$ shines down on the sphere of radius 1 centred at $(3,0,2)$ casting a shadow on the xy-plane. The area of the shadow is greater than 33.

Solution: This is TRUE. Draw a picture to represent the situation in the xz-plane (with the y-axis pointing directly out of the page). so the light is at $(0,4)$ and the sphere is represented by a circle of radius 1 centred at $(3,2)$. Draw a right-angled triangle with vertices at $a=(0,4)$, $b=(3,2)$ and the point c along the upper half of the circle such that bc is a radius of the circle and ca is a tangent to the circle at c. The triangle has side lengths $|a-b|=\sqrt{13}$, $|b-c|=1$ and $|c-a|=\sqrt{12}$ and so that angle θ at vertex a is given by $\cos\theta = \frac{\sqrt{12}}{\sqrt{13}}$. The cone of rays of light from $(0,0,4)$ tangent to the sphere is the set of points (x,y,z) such that when u is the vector from $(0,0,4)$ to $(3,0,2)$ and v is the vector from $(0,0,4)$ to (x,y,z) we have

$$\frac{\sqrt{12}}{\sqrt{13}} = \cos\theta = \frac{u \bullet v}{|u|\,|v|} = \frac{(3,0,-2)\bullet(x,y,z-4)}{\sqrt{13}\sqrt{x^2+y^2+(z-4)^2}}.$$

The shadow is enclosed by the curve of intersection of this cone with the xy-plane, so we square both sides and set $z=0$ to get $12(x^2+y^2+16) = (3x+8)^2 = 9x^2+48x+64$, that is $3x^2-48x+12y^2+8\cdot 16 = 0$. Complete the square to get $3\big((x-8)^2-64\big)+12y^2+8\cdot16=0$, that is $3(x-8)^2+12y^2=64$. Thus the shadow is enclosed by the ellipse $\frac{(x-8)^2}{64/3}+\frac{y^2}{64/12}=1$ which has area $A=\pi\sqrt{\frac{64}{3}}\sqrt{\frac{64}{12}} = \frac{64\pi}{6} = \frac{32\pi}{3}$. Finally, note that $\frac{32\pi}{3} > \frac{(32)(3.12)}{3} = (32)(1.04) = 33.28$

9: There exists a continuous function $f:[0,1]\to[0,1]$ such that for every $y\in[0,1]$ the number of $x\in[0,1]$ for which $f(x)=y$ is finite and even.

Solution: This is TRUE. For example, we can choose a sequence $\{a_n\}$ with $0<a_1<a_2<a_3<\cdots$ such that $a_n\to 1$ and a sequence $\{b_n\}$ with $1>b_1>b_2>b_3>\cdots$ such that $b_n\to 0$, and then take $f:[0,1]\to[0,1]$ to be the function with $f(1)=0$ whose graph is the polygonal path with vertices at

$$(0,0)\,,\,(a_1,1)\,,\,(a_2,b_1)\,,\,(a_3,1)\,,\,(a_4,b_2)\,,\,(a_5,b_1)\,,\,(a_6,b_3)\,,\,(a_7,b_2)\,,\,\cdots\,,\,(a_{2k},b_k)\,,\,(a_{2k+1},b_{k-1})\,,\cdots$$

10: There exists a polynomial $f\in\mathbf{Q}[x,y]$ such that the map $f:\mathbf{N}\times\mathbf{N}\to\mathbf{N}$ is bijective.

Solution: This is TRUE. One bijection $g:\mathbf{N}\to\mathbf{N}\times\mathbf{N}$ is given by

n	0	1	2	3	4	5	6	7	8	9	10	$\cdots$
$g(n)$	$(0,0)$	$(1,0)$	$(0,1)$	$(2,0)$	$(1,1)$	$(0,2)$	$(3,0)$	$(2,1)$	$(1,2)$	$(0,3)$	$(4,0)$	$\cdots$

Let $f:\mathbf{N}\times\mathbf{N}\to\mathbf{N}$ be the inverse map. The f satisfies $f(n,0)=\frac{n(n+1)}{2}$ and is given by the formula $f(n-l,l) = f\big((n,0)+l(-1,1)\big) = f(n,0)+l = \frac{n(n+1)}{2}+l$. Letting $k=n-l$ gives $f(k,l)=\frac{(k+l)(k+l+1)}{2}+l$.

11: There exists a bijective map $f : \mathbf{Z}^+ \to [0,1] \cap \mathbf{Q}$ such that $\sum_{n=1}^{\infty} \frac{f(n)}{n}$ converges.

Solution: This is TRUE. For example, we can construct such a bijective map as follows. Let S be the set of reciprocals of positive integers, that is let $S = \left\{ \frac{1}{k} \middle| k \in \mathbf{Z}^+ \right\}$ and let T be the complement of S in $[0,1] \cap \mathbf{Q}$. Note that T is countable, and say $T = \{t_1, t_2, t_3, \cdots\}$. For $l \in \mathbf{Z}^+$, let a_l denote the l^{th} non-square so that, for example, we have $a_1 = 2$, $a_2 = 3$, $a_3 = 5$, $a_4 = 6$, $a_5 = 7$, $a_6 = 8$ and $a_7 = 10$. For $n \in \mathbf{Z}^+$, if $n = k^2$ (so that n is the k^{th} square) then let $f(n) = t_k \in T$ and if $n = a_l$ (so that n is the l^{th} non-square) then let $f(n) = \frac{1}{l} \in S$ so that, for example, we have

n	1	2	3	4	5	6	7	8	9	10
$f(n)$	t_1	$\frac{1}{1}$	$\frac{1}{2}$	t_2	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	t_3	$\frac{1}{7}$

Then since $0 \le t_k \le 1$ and $a_l \ge l$ for all $k, l \in \mathbf{Z}^+$, we have

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} = \sum_{n \text{ square}} \frac{f(n)}{n} + \sum_{n \text{ non-square}} \frac{f(n)}{n} = \sum_{k=1}^{\infty} \frac{t_k}{k^2} + \sum_{l=1}^{\infty} \frac{1/l}{a_l} \le \sum_{k=1}^{\infty} \frac{1}{k^2} + \sum_{l=1}^{\infty} \frac{1}{l^2},$$

which is finite.

12: For every sequence of real numbers $\{a_n\}$, if $\sum_{n=1}^{\infty} a_n$ converges then so does the series

$$a_1,\ a_2,\ a_4, a_3,\ a_8, a_7, a_6, a_5,\ a_{16}, a_{15}, \cdots, a_9,\ a_{32}, a_{31}, \cdots, a_{17}, a_{64}, \cdots$$

Solution: This is TRUE. Let $S = \sum_{n=1}^{\infty} a_n$. For each $m \in \mathbf{Z}^+$ let $S_m = \sum_{n=1}^{m} a_n$, and let T_m be the m^{th} partial sum for the rearranged series. Let $\epsilon > 0$. Choose $M > 0$ so that

$$m \ge M \Longrightarrow \left|S_m - S\right| < \tfrac{\epsilon}{3}.$$

Note that for $2^{n-1} \le m < 2^n$, say $m = 2^{n-1} + k$ with $0 \le k < 2^{n-1}$, we have

$$T_m = S_{2^{n-1}} + a_{2^n} + a_{2^n-1} + \cdots + a_{2^n-k+1} = S_{2^{n-1}} + S_{2^n} - S_{2^n-k}$$

and so when $2^{n-1} \ge M$ we have

$$\left|T_m - S\right| = \left|(S_{2^{n-1}} - S) + (S_{2^n} - S) - (S_{2^n-k} - S)\right| \le \left|S_{2^{n-1}} - S\right| + \left|S_{2^n} - S\right| + \left|S_{2^n-k} - S\right| < \epsilon\,.$$

13: Initially, $n = 2$. Two players, A and B, take turns with A going first. At each turn, the player whose turn it is can either replace n by $n+1$ or by $2n$. The first player to replace n by a number larger than 130 loses. In this game, player A has a winning strategy.

Solution: This is FALSE. It is player B that has the winning strategy. A player who receives $n = 130$ will lose. A player who receives $n = 129$ will win by replacing n by $n+1$. A player who receives $n = 128$ must replace n by $n+1$ and will then lose. Continuing, we see that a player who receives an odd value of n with $65 \le n \le 129$ will win by replacing n by $n+1$, while a player who receives an even value of n with $66 \le n \le 130$ will lose. Continuing further, we see that a player who receives any value of n with $33 \le n \le 65$ will win by replacing n by $2n$ (so that the opponent receives an even value of n with $66 \le n \le 130$). A player who receives an odd value of n with $17 \le n \le 31$ wins by replacing n by $n+1$, while a player who receives an even value of n with $18 \le n \le 32$ will lose (since replacing n by $n+1$ yields an odd value between 19 and 33, and replacing n by $2n$ yields a value between 33 and 65). We continue such reasoning and summarize the results in the following table.

Win:		129		$\cdots$		65	$\cdots$	33		31		$\cdots$		17	$\cdots$	9		7		5	4	3	
Lose:	130		$\cdots$		66				32		$\cdots$		18				8		6				2

We see that player A initially receives $n = 2$ so he is in a losing position. He must either replace n by 3 or by 4, placing B in a winning position. Then B can win using the following strategy. If he receives $n = 3, 4$ he replaces n by $2n$; if he receives $n = 5, 7$ he replaces n by $n+1$; if he receives $n = 9, 10, 11, \cdots, 16$ he replaces n by $2n$; if he receives $n = 17, 19, 21, \cdots, 31$ he replaces n by $n+1$; if he receives $n = 33, 34, 35, \cdots, 64$ he replaces n by $2n$; and if he receives $n = 65, 67, 69, \cdots, 129$ he replaces n by $n+1$.