Solutions to the Bernoulli Trials Problems for 2014

1: We have $\dfrac{(n^2)!}{(n!)^{n+1}} \in \mathbf{Z}$ for all $n \in \mathbf{Z}^+$.

Solution: This is TRUE. Indeed for $1 \le k \le n$ we have $\frac{1}{k}\binom{kn}{n} = \frac{(kn)(kn-1)(kn-2)\cdots(kn-n+1)}{kn(n-1)(n-2)\cdots(2)(1)} = \binom{kn-1}{n-1} \in \mathbf{Z}$ and so

$$\begin{aligned}\frac{(n^2)!}{(n!)^{n+1}} &= \frac{1}{n!}\cdot\frac{(n^2)(n^2-1)\cdots(n^2-n+1)}{n!}\cdot\frac{(n^2-n)(n^2-n-1)\cdots(n^2-2n+1)}{n!}\cdot\,\cdots\,\cdot\frac{(2n)(2n-1)\cdots(n+1)}{n!}\cdot\frac{(n)(n-1)\cdots(1)}{n!}\\ &= \frac{1}{n!}\cdot\binom{n^2}{n}\cdot\binom{n^2-n}{n}\cdot\,\cdots\,\cdot\binom{2n}{n}\cdot\binom{n}{n} = \prod_{k=1}^{n}\frac{1}{k}\binom{kn}{n} \in \mathbf{Z}\end{aligned}$$

2: There exists a right-angled triangle whose three side lengths are Fibonacci numbers.

Solution: This is FALSE. A right-angled triangle with integral sides must have all three sides of different lengths, and no three distinct Fibonacci numbers can be the side lengths of any triangle since if a, b and c are Fibonacci numbers with $a < b < c$ then we have $c \ge a + b$.

3: For all $a, b, c, d \in \mathbf{R}$ with $a \ne 0$, the cubic polynomial $f(x) = ax^3 + bx^2 + cx + d$ has one real root and two purely imaginary roots if and only if $ad = bc$ and $ac > 0$.

Solution: This is TRUE. If $f(x) = ax^3 + bx^2 + cx + d$ has one real root, say α, and two purely imaginary roots, say $\pm i\beta$ with $\beta > 0$, then we have

$$ax^3 + bx^2 + cx + d = a(x-\alpha)(x^2 - i\beta)(x + i\beta) = a(x-\alpha)(x^2+\beta^2) = ax^3 - a\alpha x^2 + a\beta^2 x - a\alpha\beta^2$$

so that $b = -a\alpha$, $c = a\beta^2$ and $d = -a\alpha\beta^2$, and hence $ad = -a^2\alpha\beta^2 = bc$ and $ac = a^2\beta^2 > 0$. Conversely, if $ad = bc$ and $ac > 0$ then we have $f(x) = ax^3 + bx^2 + cx + d = a\left(x + \frac{b}{a}\right)\left(x^2 + \frac{c}{a}\right)$.

4: There exists a polynomial $f(x)$ over $\mathbf{R}$ such that $f\left(\frac{1}{n}\right) = \frac{n+2}{n}$ for all $n \in \mathbf{Z}^+$.

Solution: This is TRUE. Indeed if $f(x) = 2x + 1$ then $f\left(\frac{1}{n}\right) = \frac{2}{n} + 1 = \frac{n+2}{n}$.

5: For every polynomial $f(x)$ over $\mathbf{Z}_5$ and for every $\alpha \in \mathbf{Z}_5$ and $m \in \mathbf{Z}^+$, if α is a root of $f(x)$ of multiplicity m then α is a root of $f'(x)$ of multiplicity $m - 1$.

Solution: This is FALSE. If $f(x) = (x-1)^5(2x^3 - x^2 + x)$ then $f'(x) = (x-1)^5(x^2 - 2x + 1) = (x-1)^7$ so that $\alpha = 1$ is a root of $f(x)$ of multiplicity 5 and a root of $f'(x)$ of multiplicity 7.

6: For every convex set $S \subseteq \mathbf{R}^3$ there exists a countable set $C \subseteq S$ such that S is the smallest convex set containing C.

Solution: This is FALSE. For example, let S be the closed unit ball $S = \left\{x \in \mathbf{R}^3 \middle| |x| \le 1\right\}$. Let $C \subseteq S$ and suppose that S is the smallest convex set which contains C. Let $x \in S$ with $|x| = 1$. Then we must have $x \in C$, otherwise C would be contained in the convex set $S \setminus \{x\}$ so that S would not be the smallest convex set containing C. This shows that C must contain every point $x \in S$ with $|x| = 1$, and so C is uncountable.

7: Let S denote the set of lines in $\mathbf{R}^3$ which do not pass through the origin and which are not parallel to any of the three coordinate planes. For a point $p = (x, y, z) \in \mathbf{R}^3$, let $|p|_2 = \sqrt{x^2 + y^2 + z^2}$ and let $|p|_\infty = \max\left\{|x|, |y|, |z|\right\}$. For a line $L \in S$, let $p(L)$ be the point $p \in L$ which minimizes $|p|_2$ and let $q(L)$ be the point $q \in L$ which minimizes $|q|_\infty$. Then for every line $L \in S$, the points $p(L)$ and $q(L)$ lie in the same octant.

Solution: This is FALSE. For example, the line L through the points $p = (1, 3, 8)$ and $q = (-3, 7, 7)$ has $p(L) = p$ and $q(L) = q$.

8: For every differentiable function $f:\mathbf{R}^+\to\mathbf{R}^+$, if $\lim\limits_{x\to 0^+} f(x)=\infty$ then $\lim\limits_{x\to 0^+} f'(x)=-\infty$.

Solution: This is FALSE. For example, let $f(x)=\frac{1}{x}+\sin\frac{1}{x}$. so that $f'(x)=-\frac{1}{x^2}\left(1+\cos\frac{1}{x}\right)$. Note that $\lim\limits_{x\to 0^+} f(x)=\infty$. But for $k\in\mathbf{Z}^+$ we have $f'\left(\frac{1}{(2k+1)\pi}\right)=0$ so that $\lim\limits_{x\to 0^+} f'(x)\neq -\infty$.

9: Define $f:\mathbf{R}\to\mathbf{R}$ by $f(x)=\displaystyle\int_0^x \cos\tfrac{1}{t}\,dt$. Then f is differentiable at $x=0$.

Solution: This is TRUE, indeed we shall show that $f'(0)=0$. Integrating by parts, using $u=t^2$ and $v=-\sin\frac{1}{t}$, we have

$$\begin{aligned}\frac{f(x)-f(0)}{x-0}&=\frac{f(x)}{x}=\frac{1}{x}\int_0^x\cos\tfrac{1}{t}\,dt=\frac{1}{x}\int_0^x t^2\cdot\tfrac{1}{t^2}\cos\tfrac{1}{t}\,dt=\frac{1}{x}\int_0^x u\,dv=\frac{1}{x}\left[uv-\int v\,du\right]_0^x\\&=\frac{1}{x}\left[-t^2\sin\tfrac{1}{t}+\int 2t\sin\tfrac{1}{t}\,dt\right]_0^x=-x\,\sin\tfrac{1}{x}+\frac{1}{x}\int_0^x 2t\sin\tfrac{1}{t}\,dt\end{aligned}$$

Since $\left|x\sin\frac{1}{x}\right|\leq|x|\to 0$ as $x\to 0$ and $\left|\dfrac{1}{x}\displaystyle\int_0^x 2t\sin\tfrac{1}{t}\,dt\right|\leq\dfrac{1}{|x|}\displaystyle\int_0^x 2t\,dt=|x|\to 0$ as $x\to 0$ we see that $f'(0)=\lim\limits_{x\to 0}\dfrac{f(x)-f(0)}{x-0}=0$.

10: $\displaystyle\lim_{n\to\infty}\prod_{k=1}^n\frac{2k-1}{2k}=0$.

Solution: This is TRUE. Let $c_n=\prod\limits_{k=1}^n\frac{2k-1}{2k}=\frac{1\cdot3\cdot5\cdots(2n-1)}{2^n n!}$. By the Binomial Theorem, for all x in the interval of convergence of the power series we have

$$(1+x)^{-1/2}=1-\tfrac{1}{2}x+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}x^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}x^3+\cdots=\sum_{n=0}^\infty(-1)^n c_n x^n.$$

By Abel's Theorem, the power series converges when $x=1$ $\left(\text{with }\sum\limits_{n=0}^\infty(-1)^n c_n=\frac{1}{\sqrt{2}}\right)$ and hence $\lim\limits_{n\to\infty}c_n=0$.

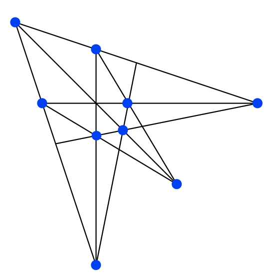
11: There exists a commutative binary operation $*$ on the set $S=\{1,2,3\}$ with the property that $x*(x*y)=y$ for all $x,y\in S$.

Solution: This is TRUE. For example, we can take $*$ to be the following operation.

$x\backslash y$	1	2	3
1	1	3	2
2	3	2	1
3	2	1	3

12: There exists a set of nine points P and a set of nine lines L in the Euclidean plane such that each point in P lies on exactly three of the lines in L, and each line in L contains exactly three of the points in P.

Solution: This is TRUE. For example we can take P and L as shown below.



13: There exists a finite set of at least two points P in the Euclidean plane with the property that the perpendicular bisector of each pair of points in P contains exactly two of the points in P.

Solution: This is TRUE. For example we can take P to be the 4 vertices of a square together with the 4 additional vertices used to erect an external equilateral triangle on each edge. To be explicit, we can take

$$P = \left\{ \pm(1,1),\, \pm(1,-1),\, \pm\big(0, 1+\sqrt{3}\big),\, \pm\big(1+\sqrt{3}, 0\big) \right\}.$$

14: It is possible to design a pair of weighted six-sided dice such that that when the dice are rolled, there is an equal probability of obtaining each of the possible sums from 2 to 12.

Solution: This is FALSE. Suppose we have designed a pair of weighted dice. For $1 \leq i \leq 6$, let p_i be the probability that the first die shows the number i and let q_i be the probability that the second die shows the number i. For $2 \leq j \leq 12$, let s_j be the probability that when we roll both dice the sum is equal to j. Suppose that $s_2 = s_{12}$. Note that we must have $(p_1 - p_6)(q_1 - q_6) \leq 0$ because if we had $(p_1 - p_6)(q_1 - q_6) > 0$ then either we would have $p_1 > p_6$ and $q_1 > q_6$ in which case $s_2 = p_1 q_1 > p_6 q_6 = s_{12}$ or we would have $p_1 < p_6$ and $q_1 < q_6$ in which case $s_2 = p_1 q_1 < p_6 q_6$. It follows that

$$s_2 + s_{12} = p_1 q_1 + p_6 q_6 = p_1 q_6 + p_6 q_1 + (p_1 - p_6)(q_1 - q_6) \leq p_1 q_6 + p_6 q_1 < s_7.$$