Bernoulli Trials Problems for 2014

1: We have $\dfrac{(n^2)!}{(n!)^{n+1}} \in \mathbf{Z}$ for all $n \in \mathbf{Z}^+$.

2: There exists a right-angled triangle whose three side lengths are Fibonacci numbers.

3: For all $a, b, c, d \in \mathbf{R}$ with $a \neq 0$, the cubic polynomial $f(x) = ax^3 + bx^2 + cx + d$ has one real root and two purely imaginary roots if and only if $ad = bc$ and $ac > 0$.

4: There exists a polynomial $f(x)$ over $\mathbf{R}$ such that $f\left(\frac{1}{n}\right) = \frac{n+2}{n}$ for all $n \in \mathbf{Z}^+$.

5: For every polynomial $f(x)$ over $\mathbf{Z}_5$ and for every $\alpha \in \mathbf{Z}_5$ and $m \in \mathbf{Z}^+$, if α is a root of $f(x)$ of multiplicity m then α is a root of $f'(x)$ of multiplicity $m - 1$.

6: For every convex set $S \subseteq \mathbf{R}^3$ there exists a countable set $C \subseteq S$ such that S is the smallest convex set containing C.

7: Let S denote the set of lines in $\mathbf{R}^3$ which do not pass through the origin and which are not parallel to any of the three coordinate planes. For a point $p = (x, y, z) \in \mathbf{R}^3$, let $|p|_2 = \sqrt{x^2 + y^2 + z^2}$ and let $|p|_\infty = \max\big\{|x|, |y|, |z|\big\}$. For a line $L \in S$, let $p(L)$ be the point $p \in L$ which minimizes $|p|_2$ and let $q(L)$ be the point $q \in L$ which minimizes $|q|_\infty$. Then for every line $L \in S$, the points $p(L)$ and $q(L)$ lie in the same octant.

8: For every differentiable function $f : \mathbf{R}^+ \to \mathbf{R}^+$, if $\lim\limits_{x \to 0^+} f(x) = \infty$ then $\lim\limits_{x \to 0^+} f'(x) = -\infty$.

9: Define $f : \mathbf{R} \to \mathbf{R}$ by $f(x) = \displaystyle\int_0^x \cos \tfrac{1}{t}\, dt$. Then f is differentiable at $x = 0$.

10: $\displaystyle\lim_{n \to \infty} \prod_{k=1}^{n} \frac{2k-1}{2k} = 0$.

11: There exists a commutative binary operation $*$ on the set $S = \{1, 2, 3\}$ with the property that $x * (x * y) = y$ for all $x, y \in S$.

12: There exists a set of nine points P and a set of nine lines L in the Euclidean plane such that each point in P lies on exactly three of the lines in L, and each line in L contains exactly three of the points in P.

13: There exists a finite set of at least two points P in the Euclidean plane with the property that the perpendicular bisector of each pair of points in P contains exactly two of the points in P.

14: It is possible to design a pair of weighted six-sided dice such that that when the dice are rolled, there is an equal probability of obtaining each of the possible sums from 2 to 12.