Bernoulli Trials Problems for 2012

1: There exists a positive integer n such that $n^3 + (n+1)^3 = (n+2)^3$.

2: There exists a positive integer n such that neither n nor n^2 uses the digit 1 in its base 3 representation.

3: For every positive integer n, n is prime if and only if there exist unique positive integers a and b such that $\frac{1}{n} = \frac{1}{a} - \frac{1}{b}$.

4: $\sqrt{1+\sqrt{7+\sqrt{1+\sqrt{7+\cdots}}}}$ is rational.

5: $\sin(20^\circ)\sin(40^\circ)\sin(60^\circ)\sin(80^\circ)$ is rational.

6: $\left(\frac{e}{2}\right)^{\sqrt{3}} < \left(\sqrt{2}\right)^{\pi/2}$.

7: Given $a \in \mathbf{R}$, let $x_1 = a$ and for $n \geq 1$ let $x_{n+1} = x_n \cos(x_n)$. Then $\{x_n\}$ converges for all choices of $a \in \mathbf{R}$.

8: Define a bijection $f : \mathbf{Z}^+ \to \mathbf{Z}^2$ by counting the elements in $\mathbf{Z}^2$ as follows. Let $f(1) = (0,0)$ and $f(2) = (1,0)$, and then continue counting by spiralling counterclockwise so that for example we have

k	1	2	3	4	5	6	7	8	9	10
$f(k)$	$(0,0)$	$(1,0)$	$(1,-1)$	$(0,-1)$	$(-1,-1)$	$(-1,0)$	$(-1,1)$	$(0,1)$	$(1,1)$	$(2,1)$

Then there exists $a \in \mathbf{Z}^+$ such that $f^{-1}(a,0)$ is a multiple of 5.

9: There exists a permutation $\{a_1, a_2, \cdots, a_{20}\}$ of the set $\{1, 2, \cdots, 20\}$ such that for all k with $1 < k < 20$, either $a_k = a_{k+1} + a_{k-1}$ or $a_k = |a_{k+1} - a_{k-1}|$.

10: There exists a permutation $\{a_1, a_2, \cdots, a_{20}\}$ of the set $\{1, 2, \cdots, 20\}$ such that for all k with $1 \leq k \leq 20$, $k + a_k$ is a power of 2.

11: There exists a partition of $\{1, 2, \cdots, 15\}$ into 5 disjoint 3-element sets $S_k = \{a_k, b_k, c_k\}$ such that $a_k + b_k = c_k$ for $k = 1, 2, 3, 4, 5$.

12: For every finite set of integers S, $\left|\left\{(a,b) \in S^2 \middle| a-b \text{ is odd}\right\}\right| \leq \left|\left\{(a,b) \in S^2 \middle| a-b \text{ is even}\right\}\right|$.

13: For every set S, whose elements are finite subsets of $\mathbf{Z}$, with the property that $A \cap B \neq \emptyset$ for all $A, B \in S$, there exists a finite set $C \subset \mathbf{Z}$ such that $A \cap B \cap C \neq \emptyset$ for all $A, B \in S$.

14: There exists a linearly independent set $\{A_1, A_2, A_3\}$ of real 3×3 matrices such that every non-zero matrix in $\text{Span}\{A_1, A_2, A_3\}$ is invertible.

15: For all 2×2 real matrices A, B and C, $\det \begin{pmatrix} I & A \\ B & C \end{pmatrix} = 0$ if and only if $\det \begin{pmatrix} I & B \\ A & C \end{pmatrix} = 0$.

16: There exists a positive integer n and an $n \times n$ matrix A whose entries lie in $\{0, 1\}$, such that $\det(A) > n$.

17: For every function $f : \mathbf{R} \to \mathbf{R}$, if f^2 and f^3 are both polynomials, then so is f.

18: Every real polynomial is equal to the difference of two increasing polynomials.

19: For every polynomial f with integer coefficients, and for all distinct integers $a_1, a_2, \cdots, a_l$, there exists an integer c such that the product $p(a_1)p(a_2)\cdots p(a_l)$ divides $f(c)$.

20: For all increasing functions $f, g : \mathbf{R} \to \mathbf{R}$ with $f(x) < g(x)$ for all $x \in \mathbf{Q}$, we have $f(x) \leq g(x)$ for all $x \in \mathbf{R}$.

21: There exists a continuously differentiable function $f : \mathbf{R} \to \mathbf{R}^+$ such that $f'(x) = f\big(f(x)\big)$ for all $x \in \mathbf{R}$.