Bernoulli Trials Problems for 2011

1: There exists a positive integer n such that for every integer a with $1{,}000 \le a \le 1{,}000{,}000$, a is prime if and only if $\gcd(a, n) = 1$.

2: For all irrational numbers x and y such that y is not a rational multiple of x, the set $\big\{\big(\langle tx\rangle, \langle ty\rangle\big)\big| t \in \mathbf{Z}\big\}$ is dense in the unit square $[0,1]\times[0,1]$. (Here $\langle x\rangle$ denotes the *fractional part* of x, that is $\langle x\rangle = x - \lfloor x\rfloor$).

3: For every positive integer n, the number of ordered pairs of positive integers (a, b) with $\text{lcm}(a, b) = n$ is equal to the number of positive divisors of n^2.

4: The last non-zero digit of 100! is equal to 4.

5: For every integer $n > 1$, n is prime if and only if $\sin\left(\dfrac{1 + (n-1)!}{n}\,\pi\right) = 0$.

6: Every periodic function $f : \mathbf{R} \to \mathbf{R}$ has a unique smallest positive period. (A function $f : \mathbf{R} \to \mathbf{R}$ is called *periodic* with *period* $p > 0$ when $f(x + p) = f(x)$ for all $x \in \mathbf{R}$).

7: There is a parabola which is tangent to every line whose x and y-intercepts add up to 1.

8: Let $a_1 = a_2 = a_3 = 1$ and let $a_n = \dfrac{1 + a_{n-1}a_{n-2}}{a_{n-3}}$ for $n \ge 4$. Then each a_n is an integer.

9: At each point $(a, b) \in \mathbf{Z}^2 \setminus \{(0, 0)\}$, there is a cylinder of height 1 whose base is a circle of radius $\frac{3}{10}$ centered at (a, b). Exactly 24 of these cylinders can be seen from the origin.

10: Let S be the unit circle $x^2 + y^2 = 1$ and let T be the unit circle with the point $(1, 0)$ removed. Then T can be partitioned into two disjoint non-empty sets A and B such that for some rotation R about the origin, the sets A and $R(B)$ form a partition of S.

11: The entries of an $n \times n$ matrix A are chosen at random from $\{1, 2, 3, \cdots, 100\}$. Let P_n be the probability that $\det(A)$ is odd. Then $0 < \lim\limits_{n\to\infty} P_n < \frac{1}{2}$.

12: For every function $f : [0, 1] \to [0, 1]$ which is continuous and non-decreasing, the length of the graph of f is less than 2. (The *length* of the graph of f is the supremum, over all partitions $0 = x_0 < x_1 < \cdots < x_n = 1$, of the sum $\sum\limits_{i=1}^{n} \sqrt{(\Delta_i x)^2 + (\Delta_i y)^2}$ where $\Delta_i x = x_i - x_{i-1}$ and $\Delta_i y = f(x_i) - f(x_{i-1})$).

13: A permutation σ of $\{1, 2, 3, \cdots, 200\}$ is chosen at random. The probability that σ contains a cycle of length exactly 100 is less than 1%.

14: For every positive integer n there exists a binary string $s = a_1a_2\cdots a_l$ of length $l = 2^n + n - 1$ such that each of the 2^n binary strings of length n occurs as a substring of s.

15: Let $a_1 = 2$ and for $n \ge 1$ let $a_{n+1} = \dfrac{a_n(n + a_n)}{n + 1}$. Then each a_n is an integer.