

## Solutions to the Bernoulli Trials Problems, 2009

- 1:** Some positive integral power of 3 ends with the digits 0001.

Solution: This is TRUE. Indeed  $\gcd(3, 10000) = 1$  so  $3^{\phi(10000)} \equiv 1 \pmod{10000}$ .

- 2:** The numbers 1, 2, 3,  $\dots$ , 2009 can be rearranged and then written one after the other in this new order to produce a single number which is a perfect cube.

Solution: This is FALSE. No matter how they are rearranged, the sum of the digits of the resulting number modulo 9 is  $S \equiv 1 + 2 + \dots + 2009 \equiv \frac{2009 \cdot 2010}{2} \equiv 2009 \cdot 1005 \equiv 2 \cdot 6 \equiv 3$ . Thus the number is a multiple of 3 but not a multiple of 9, so it cannot be a perfect square or cube or any higher power.

- 3:** The sum  $\sum_{n=4}^{\infty} \binom{n}{4}^{-1}$  is rational.

Solution: This is TRUE. Indeed we have

$$\sum_{n=4}^{\infty} \binom{n}{4}^{-1} = \sum_{n=4}^{\infty} \frac{24}{n(n-1)(n-2)(n-3)} = \sum_{n=4}^{\infty} \left( -\frac{4}{n} + \frac{12}{n-1} - \frac{12}{n-2} + \frac{4}{n-3} \right)$$

and we see that each of the sums  $\sum \left( -\frac{4}{n} + \frac{4}{n-3} \right)$  and  $\sum \left( \frac{12}{n-1} - \frac{12}{n-2} \right)$  telescopes to give a rational number.

- 4:** Let  $f(x)$  be increasing, differentiable and bounded for  $x \in [0, \infty)$ . Then  $\lim_{x \rightarrow \infty} f'(x) = 0$ .

Solution: This is FALSE. We construct a counterexample. Let  $h(x)$  be a differentiable function with  $h(x) = 0$  for  $x \leq 0$ ,  $h(x) = 1$  for  $x \geq 1$  and  $h'(x) \geq 0$  for  $0 \leq x \leq 1$  (for example, take  $h(x) = 3x^2 - 2x^3$  for  $0 \leq x \leq 1$ ), then let  $g(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} h(2^n(x-n))$ . The sum converges uniformly,  $g$  is non-decreasing, differentiable, bounded above by 2, and we have  $g'(n) = 0$  and  $g'(n + \frac{1}{2^{n+1}}) = h'(\frac{1}{2}) = \frac{3}{2}$  for all  $0 \leq n \in \mathbf{Z}$ , so  $\lim_{x \rightarrow \infty} g'(x)$  does not exist. For a strictly increasing counterexample, we can use  $f(x) = g(x) + \frac{x}{x+1}$ .

- 5:** There exists an integer  $p > 3$  such that  $p$ ,  $2p + 1$  and  $4p + 1$  are all prime.

Solution: This is FALSE. Indeed if  $p \equiv 0 \pmod{3}$  then (since  $P > 3$ ) it cannot be prime, if  $p \equiv 1 \pmod{3}$  then  $2p + 1 \equiv 0 \pmod{3}$  so  $2p + 1$  is not prime, and if  $p \equiv 2 \pmod{3}$  then  $4p + 1 \equiv 0 \pmod{3}$  so  $4p + 1$  is not prime.

- 6:** There exists a positive integer  $n$  such that  $P_n + 1$  is a perfect square, where  $P_n$  is the product of the first  $n$  primes.

Solution: This is FALSE. Suppose, for a contradiction, that  $P_n + 1 = n^2$  then  $P_n = n^2 - 1 = (n-1)(n+1)$ . Since  $n-1$  and  $n+1$  differ by 2, they have the same sign. If they are both odd then  $P_n = (n-1)(n+1)$  would be odd, but  $P_n = 2 \cdot 3 \cdot \dots \cdot p_n$  which is even. If  $n-1$  and  $n+1$  are both even then  $P_n = (n-1)(n+1)$  would be a multiple of 4, but  $P_n = 2 \cdot 3 \cdot \dots \cdot p_n$  is not a multiple of 4.

- 7:** If the positive integers are written out in order, then the  $10^{10}$ th digit in the resulting infinite string is equal to 1.

Solution: This is TRUE. The number of  $k$ -digit numbers is  $9 \cdot 10^{k-1}$ . The number of digits in the sequence of  $k$ -digit numbers is  $9 \cdot 10^{k-1} \cdot k$ . The number of digits in the sequence of numbers of at most  $k$  digits is  $9 + 90 \cdot 2 + 900 \cdot 3 + \dots + 9 \cdot 10^{k-1} \cdot k = (10-1) + 2(100-10) + 3(1000-100) + \dots + k(10^k - 10^{k-1}) = -1 - 10 - 100 - \dots - 10^{k-1} + k \cdot 10^k = k \cdot 10^k - \frac{10^k - 1}{9}$ . Taking  $k = 9$  gives  $9 \cdot 10^9 - \frac{10^9 - 1}{9}$ . Since we have  $10^{10} - \left( 9 \cdot 10^9 - \frac{10^9 - 1}{9} \right) = 1, 111, 111, 111$ , we see that the  $10^{10}$ th digit is equal to the 1, 111, 111, 111<sup>st</sup> digit in the sequence of 10-digit numbers, which is equal to the 1<sup>st</sup> digit in the 111, 111, 111<sup>st</sup> 10-digit number, which is equal to 1, since there are  $10^9$  10-digit numbers that start with 1.

**8:** A slab of stone of length 3 is rolled along the positive  $x$ -axis on 4 cylindrical logs of radius  $\frac{1}{4}$ . As the stone moves forwards, the trailing log is left behind. When the front of the stone overhangs the leading log by 1 unit, the trailing log is placed under the front of the stone. Initially, the stone is between  $x = 0$  and  $x = 3$  and the centers of the 4 cylinders are at  $x = 0, 1, 2$  and  $3$ . A curious, but somewhat ill-fated worm watches the proceedings from  $x = \frac{9}{2}$ . The unfortunate worm will be squashed twice.

Solution: This is FALSE. The worm is only squashed once (the lucky fellow). Note that the slab moves twice as fast as the centres of the logs. Let  $x_i$  be the position of (the centre of) the  $i^{\text{th}}$  log. Initially, when the slab is at  $0 \leq x \leq 3$ , the logs are at  $x_1 = 0, x_2 = 1, x_3 = 2, x_4 = 3$ . Log number 1 is then left behind. When the slab reaches  $2 \leq x \leq 5$ , the logs are at  $x_1 = 0, x_2 = 2, x_3 = 3, x_4 = 4$ , and the worm is still unsquashed. Then log number 1 is moved to  $x_1 = 5$ , and log number 2 is left behind. When the slab reaches  $4 \leq x \leq 7$ , there are logs at  $x_2 = 2, x_3 = 4, x_4 = 5, x_1 = 6$  and the worm has been squashed by log number 4. Then log number 2 is moved to  $x_2 = 7$  and log number 3 is left behind. When the slab reaches  $6 \leq x \leq 9$ , the logs are at  $x_3 = 4, x_4 = 6, x_1 = 7, x_2 = 8$ . Then log number 3 is moved to  $x_3 = 9$  and all of the logs are beyond the position of the worm, so his is safe from any further squashings.

**9:** The sum  $\sum_{n=2}^{\infty} \binom{n}{2}^{-2}$  is rational.

Solution: This is FALSE. Indeed, using Partial Fractions and the well-known formula  $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ , we have

$$\begin{aligned} \sum_{n=2}^{\infty} \binom{n}{2}^{-2} &= \sum_{n=2}^{\infty} \frac{4}{n^2(n-1)^2} = \sum_{n=2}^{\infty} \left( \frac{8}{n} + \frac{4}{n^2} - \frac{8}{n-1} + \frac{4}{(n-1)^2} \right) \\ &= 8 \sum_{n=2}^{\infty} \left( \frac{1}{n} - \frac{1}{n-1} \right) + 4 \sum_{n=2}^{\infty} \frac{1}{n^2} + 4 \sum_{k=1}^{\infty} \frac{1}{k^2} \\ &= -8 + 4 \left( \frac{\pi^2}{6} - 1 \right) + 4 \cdot \frac{\pi^2}{6} = \frac{4\pi^2}{3} - 12, \end{aligned}$$

which is irrational.

**10:** A regular octadecagon (18-gon) with sides of length 1 fits inside a circle of radius 3.

Solution: This is TRUE. Imagine a regular hexagon with sides of length 3 inscribed in a circle of radius 3. Trisect each edge of the hexagon by inserting two vertices on each edge. Imagine that all 18 vertices are hinged. By moving the original 6 vertices slightly inwards and the added 12 vertices slightly outwards, we obtain a regular 18-gon which lies inside the circle.

**11:** A regular icosahedron (20 triangular faces) with edges of length 1 fits inside the unit sphere.

Solution: This is TRUE. In the cube with sides of length 2 whose vertices are at  $(\pm 1, \pm 1, \pm 1)$  we inscribe the regular icosahedron with sides of length  $2a$  whose vertices are at  $(\pm 1, 0, \pm a), (\pm a, \pm 1, 0)$  and  $(0, \pm a, \pm 1)$ . For the sides to have equal length, the distance from  $(1, 0, a)$  to  $(a, 1, 0)$  must be equal to  $2a$ , so we need

$$(a-1)^2 + 1^2 + a^2 = 4a^2 \implies 2a^2 + 2a - 2 = 0 \implies a = \frac{-1+\sqrt{5}}{2}.$$

The length of each side is  $l = 2a$  and the distance from the origin to each vertex is  $r = \sqrt{a^2 + 1}$ . Note that  $a^2 = \left( \frac{-1+\sqrt{5}}{2} \right)^2 = \frac{6-2\sqrt{5}}{4} = \frac{3-\sqrt{5}}{2}$ , so we have

$$l^2 - r^2 = 4a^2 - (a^2 + 1) = 3a^2 - 1 = \frac{3(3-\sqrt{5})-2}{2} = \frac{7-3\sqrt{5}}{2} = \frac{\sqrt{49-45}}{2} > 0.$$

Since  $l > r$  it follows that an icosahedron with sides of length  $l$  fits inside a sphere of radius  $l$ .

**12:** In any 11 month period, the Moon moves around the Sun in a simple convex path.

Solution: This is TRUE. We model the motion of the Moon around the Sun by

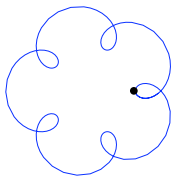
$$(x(t), y(t)) = \left( R \cos \frac{2\pi t}{P}, R \sin \frac{2\pi t}{P} \right) + \left( -r \cos \frac{2\pi t}{p}, -r \sin \frac{2\pi t}{p} \right)$$

where  $R$  is the distance from the Sun to the Earth,  $r$  is the distance from the Earth to the Moon,  $P$  is the period of the Earth's orbit around the Sun, and  $p$  is the period of the Moon's orbit around the Earth. We have

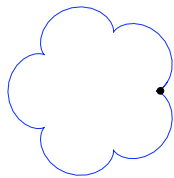
$$(x'(t), y'(t)) = \left( -\frac{2\pi R}{P} \sin \frac{2\pi t}{P}, \frac{2\pi R}{P} \cos \frac{2\pi t}{P} \right) + \left( \frac{2\pi r}{p} \sin \frac{2\pi t}{p}, -\frac{2\pi r}{p} \cos \frac{2\pi t}{p} \right)$$

$$(x''(t), y''(t)) = \left( -\frac{4\pi^2 R}{P^2} \cos \frac{2\pi t}{P}, -\frac{4\pi^2 R}{P^2} \sin \frac{2\pi t}{P} \right) + \left( \frac{4\pi^2 r}{p^2} \cos \frac{2\pi t}{p}, \frac{4\pi^2 r}{p^2} \sin \frac{2\pi t}{p} \right)$$

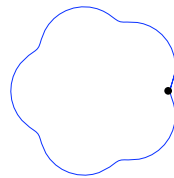
and so  $(x'(0), y'(0)) = \left( 0, 2\pi \left( \frac{R}{P} - \frac{r}{p} \right) \right)$  and  $(x''(0), y''(0)) = \left( 4\pi^2 \left( \frac{r}{p^2} - \frac{R}{P^2} \right) \right)$ . The path followed by the Moon is simple if  $y'(0) \geq 0$ , that is if  $\frac{R}{P} > \frac{r}{p}$ , and the motion will be convex if  $x''(0) < 0$ , that is if  $\frac{R}{P^2} > \frac{r}{p^2}$ . As everyone (and their dog) knows,  $R \cong 150,000,000$  km,  $r \cong 385,000$  km,  $P \cong 365$  days and  $p \cong 27.3$  days, and so we easily have  $\frac{R}{P} > \frac{r}{p}$  and  $\frac{R}{P^2} > \frac{r}{p^2}$ .



$$y'(0) < 0$$

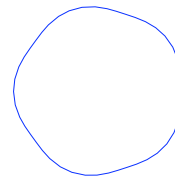


$$y'(0) = 0$$



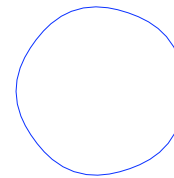
$$y'(0) > 0$$

$$x''(0) > 0$$



$$y'(0) > 0$$

$$x''(0) = 0$$



$$y'(0) > 0$$

$$x''(0) < 0$$