Solutions to the Bernoulli Trials Problems, 2009

1: Some positive integral power of 3 ends with the digits 0001.

Solution: This is TRUE. Indeed $\gcd(3, 10000) = 1$ so $3^{\phi(10000)} \equiv 1 \bmod 10000$.

2: The numbers $1, 2, 3, \cdots, 2009$ can be rearranged and then written one after the other in this new order to produce a single number which is a perfect cube.

Solution: This is FALSE. No matter how they are rearranged, the sum of the digits of the resulting number modulo 9 is $S \equiv 1 + 2 + \cdots + 2009 \equiv \frac{2009\cdot 2010}{2} \equiv 2009 \cdot 1005 \equiv 2 \cdot 6 \equiv 3$. Thus the number is a multiple of 3 but not a multiple of 9, so it cannot be a perfect square or cube or any higher power.

3: The sum $\displaystyle\sum_{n=4}^{\infty} \binom{n}{4}^{-1}$ is rational.

Solution: This is TRUE. Indeed we have

$$\sum_{n=4}^{\infty} \binom{n}{4}^{-1} = \sum_{n=4}^{\infty} \frac{24}{n(n-1)(n-2)(n-3)} = \sum_{n=4}^{\infty} \left(-\frac{4}{n} + \frac{12}{n-1} - \frac{12}{n-2} + \frac{4}{n-3} \right)$$

and we see that each of the sums $\sum \left(-\frac{4}{n} + \frac{4}{n-3} \right)$ and $\sum \left(\frac{12}{n-1} - \frac{12}{n-2} \right)$ telescopes to give a rational number.

4: Let $f(x)$ be increasing, differentiable and bounded for $x \in [0, \infty)$. Then $\lim\limits_{x\to\infty} f'(x) = 0$.

Solution: This is FALSE. We construct a counterexample. Let $h(x)$ be a differentiable function with $h(x) = 0$ for $x \le 0$, $h(x) = 1$ for $x \ge 1$ and $h'(x) \ge 0$ for $0 \le x \le 1$ (for example, take $h(x) = 3x^2 - 2x^3$ for $0 \le x \le 1$), then let $g(x) = \displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n}\, h\left(2^n(x-n)\right)$. The sum converges uniformly, g is non-decreasing, differentiable, bounded above by 2, and we have $g'(n) = 0$ and $g'\left(n + \frac{1}{2^{n+1}}\right) = h'\left(\frac{1}{2}\right) = \frac{3}{2}$ for all $0 \le n \in \mathbf{Z}$, so $\lim\limits_{x\to\infty} g'(x)$ does not exist. For a strictly increasing counterexample, we can use $f(x) = g(x) + \frac{x}{x+1}$.

5: There exists an integer $p > 3$ such that p, $2p+1$ and $4p+1$ are all prime.

Solution: This is FALSE. Indeed if $p \equiv 0 \bmod 3$ then (since $P > 3$) it cannot be prime, if $p \equiv 1 \bmod 3$ then $2p + 1 \equiv 0 \bmod 3$ so $2p = 1$ is not prime, and if $p \equiv 2 \bmod p$ then $4p + 1 \equiv 0 \bmod 3$ so $4p + 1$ is not prime.

6: There exists a positive integer n such that $P_n + 1$ is a perfect square, where P_n is the product of the first n primes.

Solution: This is FALSE. Suppose, for a contradiction, that $P_n + 1 = n^2$ then $P_n = n^2 - 1 = (n-1)(n+1)$. Since $n-1$ and $n+1$ differ by 2, they have the same sign. If they are both odd then $P_n = (n-1)(n+1)$ would be odd, but $P_n = 2 \cdot 3 \cdots p_n$ which is even. If $n-1$ and $n+1$ are both even then $P_n = (n-1)(n+1)$ would be a multiple of 4, but $P_n = 2 \cdot 3 \cdots p_n$ is not a multiple of 4.

7: If the positive integers are written out in order, then the 10^{10}th digit in the resulting infinite string is equal to 1.

Solution: This is TRUE. The number of k-digit numbers is $9 \cdot 10^{k-1}$. The number of digits in the sequence of k-digit numbers is $9 \cdot 10^{k-1} \cdot k$. The number of digits in the sequence of numbers of at most k digits is $9 + 90 \cdot 2 + 900 \cdot 3 + \cdots + 9 \cdot 10^{k-1} \cdot k = (10 - 1) + 2(100 - 10) + 3(1000 - 100) + \cdot + k(10^k - 10^{k-1}) = -1 - 10 - 100 - \cdots - 10^{k-1} + k \cdot 10^k = k \cdot 10^k - \frac{10^k - 1}{9}$. Taking $k = 9$ gives $9 \cdot 10^9 - \frac{10^9-1}{9}$. Since we have $10^{10} - \left(9 \cdot 10^9 - \frac{10^9-1}{9}\right) = 1,111,111,111$, we see that the 10^{10}th digit is equal to the $1,111,111,111^{\text{st}}$ digit in the sequence of 10-digit numbers, which is equal to the 1^{st} digit in the $111,111,111^{\text{st}}$ 10-digit number, which is equal to 1, since there are 10^9 10-digit numbers that start with 1.

8: A slab of stone of length 3 is rolled along the positive x-axis on 4 cylindrical logs of radius $\frac{1}{4}$. As the stone moves forwards, the trailing log is left behind. When the front of the stone overhangs the leading log by 1 unit, the trailing log is placed under the front of the stone. Initially, the stone is between $x = 0$ and $x = 3$ and the centers of the 4 cylinders are at $x = 0, 1, 2$ and 3. A curious, but somewhat ill-fated worm watches the proceedings from $x = \frac{9}{2}$. The unfortunate worm will be squashed twice.

Solution: This is FALSE. The worm is only squashed once (the lucky fellow). Note that the slab moves twice as fast as the centres of the logs. Let x_i be the position of (the centre of) the i^{th} log. Initially, when the slab is at $0 \le x \le 3$, the logs are at $x_1 = 0$, $x_2 = 1$, $x_3 = 2$, $x_4 = 3$. Log number 1 is then left behind. When the slab reaches $2 \le x \le 5$, the logs are at $x_1 = 0$, $x_2 = 2$, $x_3 = 3$, $x_4 = 4$, and the worm is still unsquashed. Then log number 1 is moved to $x_1 = 5$, and log number 2 is left behind. When the slab reaches $4 \le x \le 7$, there are logs at $x_2 = 2$, $x_3 = 4$, $x_4 = 5$, $x_1 = 6$ and the worm has been squashed by log number 4. Then log number 2 is moved is moved to $x_2 = 7$ and log number 3 is left behind. When the slab reaches $6 \le x \le 9$, the logs are at $x_3 = 4$, $x_4 = 6$, $x_1 = 7$, $x_2 = 8$. Then log number 3 is moved to $x_3 = 9$ and all of the logs are beyond the position of the worm, so his is safe from any further squashings.

9: The sum $\displaystyle\sum_{n=2}^{\infty} \binom{n}{2}^{-2}$ is rational.

Solution: This is FALSE. Indeed, using Partial Fractions and the well-known formula $\sum\limits_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, we have

$$\begin{aligned}\sum_{n=2}^{\infty} \binom{n}{2}^{-2} &= \sum_{n=2}^{\infty} \frac{4}{n^2(n-1)^2} = \sum_{n=2}^{\infty} \left(\tfrac{8}{n} + \tfrac{4}{n^2} - \tfrac{8}{n-1} + \tfrac{4}{(n-1)^2} \right) \\ &= 8 \sum_{n=2}^{\infty} \left(\tfrac{1}{n} - \tfrac{1}{n-1} \right) + 4 \sum_{n=2}^{\infty} \frac{1}{n^2} + 4 \sum_{k=1}^{\infty} \frac{1}{k^2} \\ &= -8 + 4 \left(\tfrac{\pi^2}{6} - 1 \right) + 4 \cdot \tfrac{\pi^2}{6} = \tfrac{4\pi^2}{3} - 12 \,,\end{aligned}$$

which is irrational.

10: A regular octadecagon (18-gon) with sides of length 1 fits inside a circle of radius 3.

Solution: This is TRUE. Imagine a regular hexagon with sides of length 3 inscribed in a circle of radius 3. Trisect each edge of the hexagon by inserting two vertices on each edge. Imagine that all 18 vertices are hinged. By moving the original 6 vertices slightly inwards and the added 12 vertices slightly outwards, we obtain a regular 18-gon which lies inside the circle.

11: A regular icosahedron (20 triangular faces) with edges of length 1 fits inside the unit sphere.

Solution: This is TRUE. In the cube with sides of length 2 whose vertices are at $(\pm 1, \pm 1 \pm 1)$ we inscribe the regular icosahedron with sides of length $2a$ whose vertices are at $(\pm 1, 0, \pm a)$, $(\pm a, \pm 1, 0)$ and $(0, \pm a, \pm 1)$. For the sides to have equal length, the distance from $(1, 0, a)$ to $(a, 1, 0)$ must be equal to $2a$, so we need

$$(a-1)^2 + 1^2 + a^2 = 4a^2 \Longrightarrow 2a^2 + 2a - 2 = 0 \Longrightarrow a = \tfrac{-1+\sqrt{5}}{2} \,.$$

The length of each side is $l = 2a$ and the distance from the origin to each vertex is $r = \sqrt{a^2 + 1}$. Note that $a^2 = \left(\frac{-1+\sqrt{5}}{2} \right)^2 = \frac{6-2\sqrt{5}}{4} = \frac{3-\sqrt{5}}{2}$, so we have

$$l^2 - r^2 = 4a^2 - (a^2 + 1) = 3a^2 - 1 = \tfrac{3(3-\sqrt{5})-2}{2} = \tfrac{7-3\sqrt{5}}{2} = \tfrac{\sqrt{49}-\sqrt{45}}{2} > 0 \,.$$

Since $l > r$ it follows that an icosahedron with sides of length l fits inside a sphere of radius l.

12: In any 11 month period, the Moon moves around the Sun in a simple convex path.

Solution: This is TRUE. We model the motion of the Moon around the Sun by

$$\big(x(t),y(t)\big) = \left(R\cos\frac{2\pi\,t}{P}, R\sin\frac{2\pi\,t}{P}\right) + \left(-r\cos\frac{2\pi\,t}{p}, -r\sin\frac{2\pi\,t}{p}\right)$$

where R is the distance from the Sun to the Earth, r is the distance from the Earth to the Moon, P is the period of the Earth's orbit around the Sun, and p is the period of the Moon's orbit around the Earth. We have

$$\big(x'(t),y'(t)\big) = \left(-\frac{2\pi\,R}{P}\sin\frac{2\pi\,t}{P}, \frac{2\pi\,R}{P}\cos\frac{2\pi\,t}{P}\right) + \left(\frac{2\pi\,r}{p}\sin\frac{2\pi\,t}{p}, -\frac{2\pi\,r}{p}\cos\frac{2\pi\,t}{p}\right)$$

$$\big(x''(t),y''(t)\big) = \left(-\frac{4\pi^2 R}{P^2}\cos\frac{2\pi\,t}{P}, -\frac{4\pi^2 R}{P^2}\sin\frac{2\pi\,t}{P}\right) + \left(\frac{4\pi^2 r}{p^2}\cos\frac{2\pi\,t}{p}, \frac{4\pi^2 r}{p^2}\sin\frac{2\pi\,t}{p}\right)$$

and so $\big(x'(0),y'(0)\big) = \left(0, 2\pi\left(\frac{R}{P}-\frac{r}{p}\right)\right)$ and $\big(x''(0),y''(0)\big) = \left(4\pi^2\left(\frac{r}{p^2}-\frac{R}{P^2}\right)\right)$. The path followed by the Moon is simple if $y'(0)\geq 0$, that is if $\frac{R}{P}>\frac{r}{p}$, and the motion will be convex if $x''(0)<0$, that is if $\frac{R}{P^2}>\frac{r}{p^2}$. As everyone (and their dog) knows, $R\cong 150,000,000$ km, $r\cong 385,000$ km, $P\cong 365$ days and $r\cong 27.3$ days, and so we easily have $\frac{R}{P}>\frac{r}{p}$ and $\frac{R}{P^2}>\frac{r}{p^2}$.