Bernouli Trials Problems, 2007

1: Suppose $n = 541G5072H6$ where G and H are single digits in base 10.
T or F: There are exactly 3 pairs of digits (G, H) such that n is divisible by 72.

2: Asafa ran at a speed of 21 km/h from P to Q to R to S, where $P = (0, 8)$, $Q = (0, 0)$, $R = (15, 0)$ and $S = (22, 0)$ (where the units are km). Florence ran at a constant speed from P directly to R and then to S. They left P at the same time and arrived at S at the same time.
T or F: Florence arrived at R exactly 7 minutes before Asafa.

3: The *weight* of a matrix is its number of non-zero entries.
T or F: The number of 15 by 16 matrices with entries from $\mathbf{Z}_{17}$ of rank 2 and weight 2 is larger than 6452100.

4: Let R_n be the region in the first quadrant bounded by $y = f(x) = x^n$ and $y = g(x) = \sqrt[n]{x}$.
T or F: As $n \to \infty$, the centroid of R_n approaches $\left(\frac{1}{e}, \frac{1}{e}\right)$.

5: T or F: There exist 803 disjoint unordered pairs of disstinct positive integers with distinct sums of at most 2007.

6: Let W be the set of finite products of rational numbers of the form $\dfrac{3n+2}{2n+1}$ with $n \geq 0$.
T or F: $5 \notin W$.

7: Define $f : [0, 1] \times [0, 1] \to [0, 1] \times [0, 1]$ by $f(x, y) = \left(\frac{1}{2}(x + y), \sqrt{xy}\right)$.
T or F: The area of the image of f is $\frac{1}{6}$.

8: T or F: There exist two non-congruent triangles of equal perimeter and equal area.

9: Let $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{n}$ be an infinite sum where $a_n = a_{n+4}$ for all $n \geq 1$.
T or F: The sum converges if and only if $a_1 + a_2 + a_3 + a_4 = 0$.

10: Suppose the 52 cards in a deck are numbered from 1 to 52 (from top to bottom). Lino then perfectly shuffles the deck 2007 times (the top 26 cards are interleaved with the bottom 26 cards, starting with a card from the bottom half).
T or F: The card numbered 42 ends up in position 34 (counting from the top).

11: Ken quickly calculates that $\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^2}{945}$. Ken never makes mistakes.

T or F: $\sum_{\substack{n \text{ is cubefree} \\ \text{with } n \geq 1}} \frac{1}{n^2} \leq 1.575.$

(A positive integer is called *cubefree* if it is not divisible by the cube of any integer greater than 1).

12: There exists a continuous function $f : [0,1] \times [0,1]$ with the property that for every $y \in [0,1]$, there exists infinitely many values of $x \in [0,1]$ such that $f(x) = y$.