Bernouli Trials Problems, 2006

1: Let $f(x) = \ln\left(\sqrt{1 + e^{\cos x}}\right)$.
T or F: $f(0)f'(0)f''(0) = -1$.

2: Let $S = 1 + 2 + 3 + \cdots + 2006$.
T or F: S has at least 4 distinct prime factors.

3: The lengths of the six edges of tetrahedron $ABCD$ are 7, 13, 18, 27, 36 and 41.
T or F: It possible that $AB = 41$ and $CD = 7$.

4: T or F: The equation $\sqrt{x + 3 - 4\sqrt{x-1}} + \sqrt{x = 8 - 6\sqrt{x-1}} = 1$ has exactly 8 real solutions.

5: Suppose $\left(1 + x + x^2\right)^{50} = a_0 + a_1 x + \cdots + a_{100} x^{100}$.
T or F: $a_0 + a_2 + a_4 + \cdots + a_{100}$ is odd.

6: Let A be the statement "14! = 87278291200". Let B be the statement "The minimum value for the sum of the x- and y-intercepts of a line with negative slope passing through $(3, 12)$ is 27. Let C be the statement "There is exactly one point on the Earth with the property that when Tom starts at that point, travels 100 km south, then 100 km east, then 100 km north, he ends up at the point where he started".
T or F: A is false, B is true, and C is true.

7: A heavy ball of radius 3 is placed into a cup which is in the shape of an inverted cone of radius 5 and height 10.
T or F: When the cup is filled to the brim with water, the ball is totally submerged.

$7\frac{1}{2}$: T or F: L'Hôpital's Rule was discovered by James Bernoulli.

8: Let $f_1 = f_2 = 1$ and for $n \geq 3$ let $f_n = f_{n-1} + f_{n-2}$.
T or F: $1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{f_k f_{k+1}} < \frac{1}{\sqrt{5}}$.

9: A positive integer n in base 10 has fewer than 25 digits and begins with the digits 15 on the left. When multiplied by 5, the only change in the number is to shift the digits 15 to the right hand end of the integer.
T or F: n has 16 digits.

10: For a real number p, define $S_k(p) = \sum_{n=1}^{k} \frac{1}{n^p}$.
T or F: If $p < 1$ then $\lim_{n \to \infty} \frac{S_{2n}(p)}{S_n(p)} = 2^{-p}$.

11: Larry and Barry are playing tennis. When Larry is serving, the probability that he wins a given point is $\frac{2}{3}$ because of his sinister techniques.
T or F: If p is the probability that Larry wins a game when he serves, then $\left|p - \frac{6}{7}\right| \geq \frac{1}{850}$.

12: T or F: The Diophantine equation $a^3 + b^3 = c^4$ has infinitely many solutions with c odd.

13: Let $p(x)$ be a monic quadratic polynomial with integral roots -1 and r.
T or F: There are exactly two values of r for which the equation $p(p(x)) = 0$ has exactly 3 distinct real solutions.

14: T or F: The limit $\lim\limits_{n\to\infty} e^{1+\sin\left(\frac{1}{1}\right)+\sin\left(\frac{1}{2}\right)+\cdots+\sin\left(\frac{1}{n}\right)} - e^{\sin\left(\frac{1}{1}\right)+\sin\left(\frac{1}{2}\right)+\cdots+\sin\left(\frac{1}{n}\right)}$ exists.

15: Define $p_n(x) = \displaystyle\sum_{k=0}^{n} \frac{x^k}{k!}$.
T or F: There does not exist a positive integer n for which $p_n(x)$ has a repeated root.

16: T or F: $\displaystyle\lim_{x\to 0} \frac{\sin(\tan x) - \tan(\sin x)}{x^7} > \frac{\pi}{24}$.