1. A long wire has a round cross section and is one millimeter in diameter. When tightly wound into a spherical ball it is two meters in diameter.
   TRUE or FALSE?
   When unravelled the wire will stretch from Waterloo to Toronto but not from Waterloo and Toronto and back to Waterloo again.

FALSE

The volume of the sphere is $\frac{4}{3}\pi$ m^3.

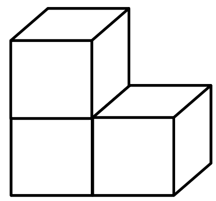
If the length of the wire is L m, then the volume of the wire is $\pi(0.0005)^2 L$ m^3.

Thus

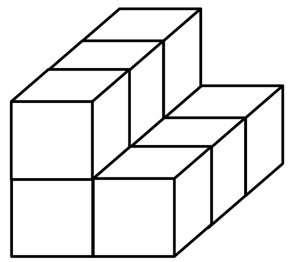
$$\begin{aligned} \frac{4}{3}\pi &= \pi(0.0005)^2 L \\ L &= (2000)^2 \frac{4}{3} \\ L &= \frac{1}{3}(16000000) \end{aligned}$$

so the length of the wire is more than 5000 km.

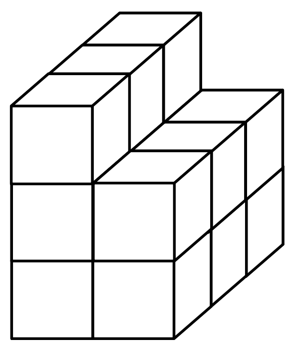
2. TRUE or FALSE?
A solid $3 \times 3 \times 3$ cube cannot be built using only pieces shaped like:



FALSE.
First, we combine 3 L's to obtain



Then we combine 6 L's to form three 3 by 2 by 1 rectangular prisms.
We put one on the bottom of the solid above, to form



We then put one of these rectangular prisms along the right face and then one along the top, to form our cube.

3. TRUE or FALSE?
   $2^{27} + 208$ is divisible by 521.

TRUE.

$$\begin{aligned} 2^{27} + 208 &= (2^9)^3 + 9^3 - 521 \\ &= (2^9 + 9)((2^9)^2 - 9(2^9) + 9^2) - 521 \\ &= (521)((2^9)^2 - 9(2^9) + 9^2 - 1) \end{aligned}$$

4. Define $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ (the Riemann zeta function).

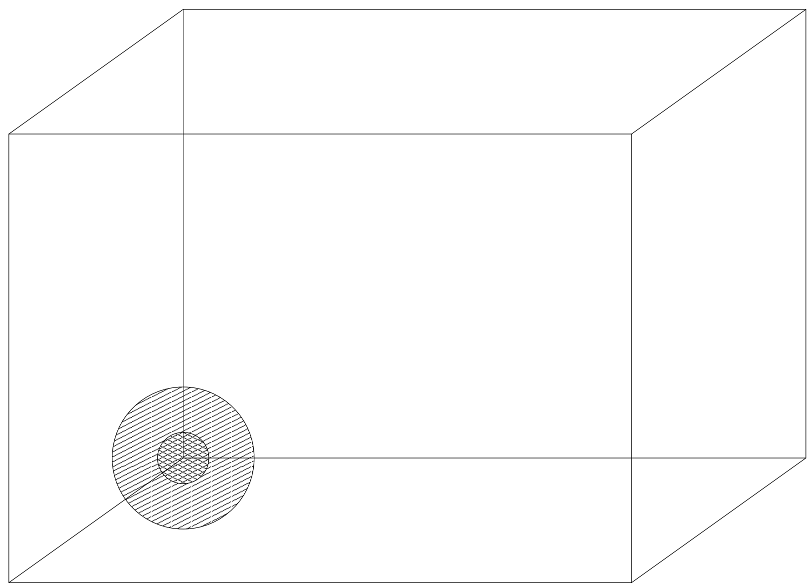
TRUE or FALSE?

$$\sum_{m=2}^{\infty} (\zeta(m) - 1) = \frac{\pi^2}{8}$$

FALSE

$$
\begin{aligned}
\sum_{m=2}^{\infty}(\zeta(m)-1) &= \sum_{m=2}^{\infty}\sum_{n=2}^{\infty} n^{-m} \\
&= \sum_{n=2}^{\infty}\sum_{m=2}^{\infty} n^{-m} \\
&= \sum_{n=2}^{\infty} \frac{\frac{1}{n^2}}{1-\frac{1}{n}} \\
&= \sum_{n=2}^{\infty} \frac{1}{n(n-1)} \\
&= \sum_{n=2}^{\infty} \left(\frac{1}{n-1}-\frac{1}{n}\right) \\
&= 1
\end{aligned}
$$

5. A sphere is placed in the corner of a rectangular box so that the sphere is tangent to the three sides which meet at the corner. A second sphere is placed between the first sphere and the corner so that it is also tangent to the three sides and also tangent to the first sphere.



Let R be the radius of the first sphere and r the radius of the second sphere.
TRUE or FALSE?
$R/r = 4$

FALSE.
Consider the distance between the centres of the spheres. This is $R+r$. However the projection of these two centres on each axis (i.e., edge of the box at the corner) gives two points a distance $R-r$ apart. So

$$(R+r)^2 = 3 \cdot (R-r)^2$$

which reduces to $R = (2+\sqrt{3})r \neq 4r$.

6. TRUE or FALSE?

$$\frac{1}{3+\frac{1}{3+\frac{1}{4+\frac{1}{5+\frac{1}{...+\frac{1}{2005}}}}}}+\frac{1}{1+\frac{1}{2+\frac{1}{3+\frac{1}{4+\frac{1}{5+\frac{1}{...+\frac{1}{2005}}}}}}}=1$$

TRUE.

Let $x = \frac{1}{3+\frac{1}{4+\frac{1}{5+\frac{1}{...+\frac{1}{2005}}}}}$.

Then the expression on the left side is

$$\frac{1}{3+x} + \frac{1}{1+\frac{1}{2+x}} = \frac{1}{3+x} + \frac{1}{\frac{3+x}{2+x}} = 1$$

7. Helen stands at the origin of the x-axis and tosses a fair die. After each toss, she takes one step in the positive direction if the die shows 1 or 2, she takes one step in the negative direction if the die shows 3 or 4, and she remains stationary if the dies shows 5 or 6.
TRUE or FALSE?
When the probability that the Helen is at the origin after 6 tosses is written as $\frac{p}{q}$ with $\gcd(p, q) = 1$, then $q - p$ is a perfect square.

TRUE.
Consider strings of length six where each letter is either P (for positive), N (for negative), or O (for origin).
We create a string for each possible sequence of six tosses.
Each letter has 3 possibilities, so there are $3^6 = 729$ possible strings.
For Helen to be at the origin, the corresponding string must have the same number of P's and N's.
The number of strings with 0 P's and 0 N's is 1.
The number of strings with 1 P and 1 N is 30.
The number of strings with 2 P's and 2 N's is 90.
The number of strings with 3 P's and 3 N's is 20.
Thus, the probability is $\dfrac{1+30+90+20}{729} = \dfrac{141}{729} = \dfrac{47}{243}$, which is in lowest terms.
Therefore, $q - p = 196 = 14^2$.

8. A positive integer n less than or equal to 10^{2005} is chosen at random.
TRUE or FALSE?
The probability that n cannot be written as a sum of three perfect squares is less than or equal to $\frac{3}{25}$.

FALSE.
No positive integer of the form $8k+7$ can be written as the sum of three perfect squares, since every perfect square is congruent to 0, 1 or 4 modulo 8.
Therefore, the probability is at least $\frac{1}{8}$.

9. Suppose $P(x)$ is a polynomial with real coefficients such that $P(x^2)$ is an integer for every integer x.
TRUE or FALSE?
$(P(x))^2$ is an integer for every integer x.

FALSE.
Try $P(x) = x(x-1)/4$. For a perfect square we get

$$P\left(x^2\right) = \frac{(x-1) \cdot x \cdot x \cdot (x+1)}{4}$$

which is an integer because the numerator has at least two factors of 2.

On the other hand $P(3) = 3/2$.

10. Two distinct numbers are chosen at random from the set $\{0, 1, 2, \ldots, 2006003\}$. Let p be the probability that the two numbers differ by a multiple of 2004.
TRUE or FALSE?

$$\left| p - \frac{1}{2004} \right| \geq \frac{1}{2006004}$$

FALSE.

Note first that $2006003 = 2004(1001) - 1$.

There are $2004(1000)$ pairs in the set which differ by 2004, $2004(999)$ which differ by $2(2004)$, and so on, with 2004 pairs which differ by $2004(1000)$. Thus,

$$p = \frac{2004 + 2004(2) + \cdots + 2004(1000)}{\frac{1}{2}(2004(1001) - 1)(2004(1001))} = \frac{1000}{2004(1001) - 1}$$

and so

$$p - \frac{1}{2004} = \frac{1000}{2004(1001) - 1} - \frac{1}{2004} = -\frac{2003}{2004(2004(1001) - 1)}$$

and

$$\frac{2003}{2004(2004(1001) - 1)} < \frac{1}{2004(1001)}$$

11. Let S denote the set of values $\alpha > 0$ such that

$$\sum_{n=1}^{\infty}(2-e^{\alpha})(2-e^{\alpha/2})\cdots(2-e^{\alpha/n})$$

converges, and let T denote the set of $\alpha > 0$ where the series diverges.

TRUE or FALSE?

Both S and T are infinite sets.

TRUE

If $\alpha = k \ln 2$ for some positive integer k, then

$$2 - e^{\alpha/k} = 2 - e^{\ln 2} = 0$$

so the terms in the series all zero for $n \geq k$.

Thus, if $\alpha = k \ln 2$, then the series converges so S is an infinite set.

Now, we all undoubtedly remember Raabe's test for divergence which says that if $\lim_{n\to\infty} n\left(1 - \left|\frac{a_{n+1}}{a_n}\right|\right) < 1$ then the series $\sum a_n$ diverges or converges conditionally.

If $0 < \alpha < 1$ with $\alpha \neq \ln 2$, then the terms in the sequence all have the same sign since $2 - e^{\alpha/k} > 0$ for $k \geq 2$, so the series cannot converge conditionally.

Also,

$$\begin{aligned}
\lim_{n\to\infty} n\left(1 - \left|\frac{a_{n+1}}{a_n}\right|\right) &= \lim_{n\to\infty} n\left(1 - \left(2 - e^{\alpha/(n+1)}\right)\right) \\
&= \lim_{n\to\infty} n\left(e^{\alpha/(n+1)} - 1\right) \\
&= \alpha \\
&< 1
\end{aligned}$$

by l'Hôpital's Rule.

Thus, the series diverges for $0 < \alpha < 1$, $\alpha \neq \ln 2$, so T is also an infinite set.

12. Let

$$f(x) = \lim_{n\to\infty} \frac{\cos^2(\pi x) + \cos^4(2\pi x) + \cdots + \cos^{2n}(n\pi x)}{n}$$

TRUE or FALSE?

$f(x)$ is continuous at $x = 0$

FALSE.

Define $f_n(x) = \dfrac{\cos^2(\pi x) + \cos^4(2\pi x) + \cdots + \cos^{2n}(n\pi x)}{n}$.

Then $f_n(0) = 1$ for all n, so $f(0) = 1$.

If $x = p/q$ is rational in lowest terms, then every qth term in the numerator of $f_n(p/q)$ is equal to 1 and the other terms will be close to 0, so $f(p/q) = 1/q$.

If x is irrational, we have an irrational winding of the circle. So most terms in the numerator are close to zero, all are between zero and one. So $f(x) = 0$.

Therefore, $f(x)$ is not continuous at $x = 0$.

13. Let a and b be positive integers, with $GCD(a, b) = 1$ and b not a prime number.
TRUE or FALSE?
There are integers $m, a_1, \ldots, a_{b-1}$ with $0 \leq a_k \leq k-1$ for $k = 1, 2, \ldots, b-1$ and

$$\frac{a}{b} = m + \sum_{k=1}^{b-1} \frac{a_k}{k!}$$

FALSE.
The equation
$$\frac{1}{4} = \frac{a_2}{2} + \frac{a_3}{6}$$
has no solution with $0 \leq a_2 \leq 1$ and $0 \leq a_3 \leq 2$. In fact, $b = 4$ is the only case for which the statement is false.