Bernouli Trials Problems, 2005

1: A long wire has a round cross section and is one millimeter in diameter. When tightly wound into a spherical ball it is two meters in diameter.
T or F: When unravelled, the wire will stretch from Waterloo to Toronto but not from Waterloo to Toronto and then back to Waterloo again.

2: T or F: A solid $3 \times 3 \times 3$ cube cannot be build using L shaped pieces made from three unit cubes.

3: T or F: $2^{27} + 208$ is divisible by 521.

4: Define $\zeta = \sum_{n=1}^{\infty} n^{-s}$.
Tor F: $\sum_{m=2}^{\infty} \big(\zeta(m) - 1\big) = \frac{\pi^2}{8}$.

5: A sphere is placed in the corner of a room so that the sphere is tangent to the three sides which meet at the corner. A second sphere is placed between the first sphere and the corner so that it is tangent to the three sides also to the first sphere. Let R be the radius of the first sphere and let r be the radius of the second sphere.
T or F: $R/r = 4$.

6: T or F: $\dfrac{1}{3 + \frac{1}{3+\frac{1}{4+\frac{1}{5+\frac{1}{\cdots+\frac{1}{2005}}}}}} + \dfrac{1}{1 + \frac{1}{2+\frac{1}{3+\frac{1}{4+\frac{1}{5+\frac{1}{\cdots+\frac{1}{2005}}}}}}} = 1.$

7: Helen stands at the origin of the x-axis and tosses a fair die. After each toss, she takes one step in the positive direction if the die shows 1 or 2, she takes one step in the negative direction if it shows 3 or 4, and she remains stationary if it shows 5 or 6.
T or F: When the probability that Helen is at the origin is written as $\frac{p}{q}$ with $\gcd(p, q) = 1$, then $q - p$ is a perfect square.

8: A positive integer n less than or equal to 10^{2005} is chosen at random.
T or F: The probability that n cannot be written as a sum of three perfect squares is equal to $\frac{3}{25}$.

9: Suppose $p(x)$ is a polynomial with real coefficients such that $p(x^2)$ is an integer for every integer x.
T or F: $p(x)^2$ is an integer for every integer x.

10: Two distinct numbers are chosen at random from the set $\{0, 1, 2, \cdots, 2006003\}$. Let p be the probability that the two numbers differ by a multiple of 2004.
T or F: $\left|p - \frac{1}{2004}\right| \geq \frac{1}{2006004}$.

11: Let S denote the set of values $\alpha > 0$ such that $\displaystyle\sum_{n=1}^{\infty} (2 - e^{\alpha}) \left(2 - e^{\alpha/2}\right) \cdots \left(2 - e^{\alpha/n}\right)$ converges, and let T denote the set of $\alpha > 0$ such that the series diverges.
T or F: Both S and T are infinite sets.

12: Let $f(x) = \displaystyle\lim_{n\to\infty} \frac{\cos^2(2\pi x) + \cos^4(\pi x) + \cdots + \cos^{2n}(n\pi x)}{n}$.
T or F: $f(x)$ is continuous at $x = 0$.

13: Let a and b be positive integers with $\gcd(a, b) = 1$ and b not a prime number.
T or F: There are integers $m, a_1, a_2, \cdots, a_{b-1}$ with $0 \leq a_k \leq k - 1$ for $k = 1, 2, \cdots, b - 1$ such that $\displaystyle\frac{a}{b} = m + \sum_{k=1}^{b-1} \frac{a_k}{k!}$.