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Area of a Chain of Circles

E 3098 [1985, 428–429]. *Proposed by Roger Cuculiere, Paris, France.*

Given two circles with diameters $IA = a$ and $IB = b$, and a set of smaller circles between them as in the following figures, find the total area enclosed by the small shaded circles in each of the following cases:

(a) The center of one of the small circles lies on the (common) diameter of the large circles (Fig. 1).

(b) Two of the small circles are tangent to the diameter of the large circles (Fig. 2).

(c) The case with no restrictions (Fig. 3).

Solution by William J. Gilbert, University of Waterloo, Canada. Let a, b and c_n be the diameters of the two large circles and the circle with center C_n. Let $\alpha = 2/a$, $\beta = 2/b$, and $\gamma_n = 2/c_n$ be their curvatures. Soddy's Theorem [H. S. M. Coxeter, *Introduction to Geometry*, pp. 13–15] states that if four circles E_1, E_2, E_3, and E_4, with curvatures ε_1, ε_2, ε_3, and ε_4, are tangent to each other then

$$(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)^2 = 2(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2),$$

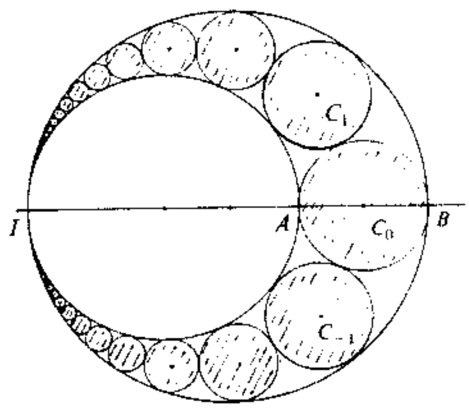

FIG. 1

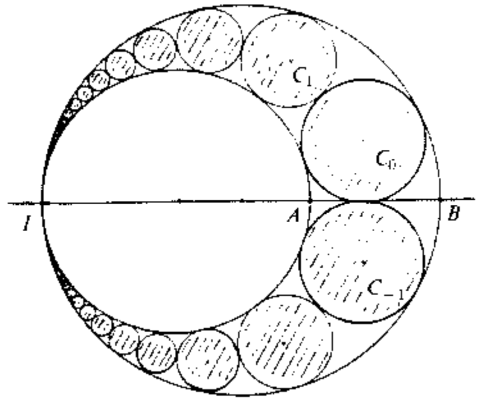

FIG 2

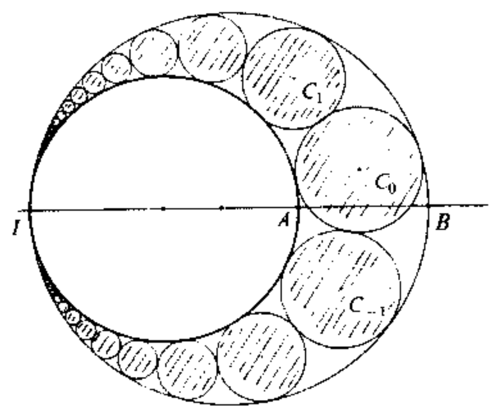

FIG. 3

where the curvature of a circle is taken to be positive if the contacts with the other circles are external and negative if the other circles are inside it. If E_2, E_3, E_4, and E_5 are also circles tangent to each other, then two applications of Soddy's Theorem yield the curvature of E_5 as $\varepsilon_5 = -\varepsilon_1 + 2\varepsilon_2 + 2\varepsilon_3 + 2\varepsilon_4$, using the same sign convention. In our case, this yields $\gamma_n - 2\gamma_{n-1} + \gamma_{n-2} = 2(\alpha - \beta)$, for all $n \in \mathbb{Z}$. The general solution to this nonhomogeneous linear difference equation is $\gamma_n = (\alpha - \beta)n^2 + Pn + Q$, where P and Q are arbitrary constants. Using γ_0 and γ_1 to evaluate these constants, we obtain $\gamma_n = (\alpha - \beta)(n^2 - n) + \gamma_1 n - \gamma_0(n - 1)$, for all $n \in \mathbb{Z}$. A further application of Soddy's Theorem shows that $\gamma_1 = \alpha - \beta + \gamma_0 + 2\sqrt{\alpha\gamma_0 - \beta\gamma_0 - \alpha\beta}$, so $\gamma_n = (\alpha - \beta)n^2 + \gamma_0 + 2n\sqrt{\alpha\gamma_0 - \beta\gamma_0 - \alpha\beta}$, for all $n \in \mathbb{Z}$. The total area covered by all the small circles is $A = \sum_{n=-\infty}^{\infty} \pi/\gamma_n^2$. In the expression for γ_n, complete the square in n and express the curvatures in terms of the diameters to obtain

$$A = \frac{\pi a^2 b^2}{4(b-a)^2} \sum_{n=-\infty}^{\infty} \frac{1}{\left[(n+s)^2 + t^2\right]^2},$$

where $t = \sqrt{ab}/(b-a)$ and $s = t\sqrt{(b-a-c_0)/c_0}$. Using contour integration [M. R. Spiegel, Schaum's *Outline of Complex Variables*, Ch. 7, Solved Problem 25], this doubly infinite sum $\Sigma f(n)$ can be computed as minus the sum of the residues of $\pi \cot \pi z\, f(z)$ at the poles of $f(z)$. Hence, computing the residues at $-s \pm it$ and simplifying, we obtain the area in the general case (c) to be

$$A = \frac{\pi^3 ab}{8}\left[\frac{\sinh y \cosh y}{y(\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y)} + \frac{\cos^2 x \sinh^2 y - \sin^2 x \cosh^2 y}{(\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y)^2}\right]$$

where $y = \pi t = \pi\sqrt{ab}/(b-a)$ and $x = \pi s = y\sqrt{(b-a-c_0)/c_0}$.

In the case (a), $c_0 = b - a$ so that $x = 0$ and the area is

$$A = \frac{\pi^3 ab}{8}\left[\frac{\coth y}{y} + \frac{1}{\sinh^2 y}\right] = \frac{\pi^3 ab}{8}\left[\frac{\coth y}{y} + \coth^2 y - 1\right].$$

In the case (b), apply Soddy's Theorem to the circles with diameters a, b, c_0 and c_{-1}, where $c_0 = c_{-1}$, to obtain $c_0 = 4ab(b-a)/(a+b)^2$. Hence, $x = \pi/2$ and the area is

$$A = \frac{\pi^3 ab}{8}\left[\frac{\tanh y}{y} - \frac{1}{\cosh^2 y}\right] = \frac{\pi^3 ab}{8}\left[\frac{\tanh y}{y} + \tanh^2 y - 1\right].$$

Also solved by T. Allen, J. R. Gosselin (Canada), H. Guggenheimer, H. Högfors, R. A. Johns, L. Kuipers (Switzerland), I. E. Leonard, O. P. Lossers (The Netherlands), J. Moisan and M. Pagès (France), W. A. Newcomb, M. Pachter (South Africa), E. Salamin, P. Zwier, and the proposer.