Positive Integer Solutions j of $\{mj/n\} \geqslant 2j/n$

E 2935* [1982, 212]. *Proposed by Clark Kimberling, University of Evansville.*

Let n be a prime greater than 10, and let $\{\quad\}$ denote fractional part. Prove or disprove that the number $S(m)$ of positive integer solutions j of the inequality $\{mj/n\} \geqslant 2j/n$ satisfies

$$S\left(\frac{n+1}{2}\right) \leqslant S(m) \leqslant S\left(\frac{n-1}{2}\right)$$

for $m = 5, 6, 7, \ldots, n-5$.

Solution by William J. Gilbert, University of Waterloo, Canada. We prove that if n is an odd prime, then:

(i) The values of $S(m)$ are symmetrical about $\frac{n}{2} + 1$, i.e.,

$$S(m) = S(n+2-m), \qquad 3 \leqslant m \leqslant n-1.$$

(ii) $S(0) = 0,\ S(1) = 0,\ S(2) = \left\lfloor \frac{n}{2} \right\rfloor,\ S(3) = \left\lfloor \frac{n}{3} \right\rfloor.$

(iii) $\left\lfloor \frac{n+1}{6} \right\rfloor = S\left(\frac{n+1}{2}\right) \leqslant S(m) \leqslant S\left(\frac{n-1}{2}\right) = \left\lfloor \frac{3n}{10} \right\rfloor$
for $4 \leqslant m \leqslant n-2$ when $n \geqslant 7$.

Here $\lfloor n \rfloor$ denotes the largest integer $\leqslant n$.

$S(m)$ is the number of points of the lattice $\left\{\left(j, \frac{m}{n}j - k\right) : j, k \in Z\right\}$ lying inside the triangle with vertices $A = (0,1)$, $B = (0,0)$ and $C = \left(\frac{n}{2}, 1\right)$ plus the number of such points lying inside the segment BC. The hypothesis that n is prime implies that there are no lattice points inside the segment BC if $m \neq 2$, since the Diophantine equation

$$(1) \qquad (2-m)x = ny$$

has no solutions with $1 \leqslant x \leqslant \frac{n}{2}$ if $m \neq 2$.

To prove (i) we make the transformation

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ (n-2m)/n & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

which takes the original lattice points into the integral points satisfying $x'' \equiv y'' \pmod 2$ and, for $m \neq 2, S(m)$ is the number of these lattice points inside the triangle $A'' = (0,2)$, $B'' = (0,0)$, $C'' = (n/2, (n-2m+4)/2)$. By symmetry in the line $y'' = 1, S(m)$ is the number of such lattice points in the triangle $A'' = (0,2)$, $B'' = (0,0)$, $D'' = (n/2, (2m-n)/2)$; hence $S(m) = S(n+2-m)$ for $3 \leqslant m \leqslant n-1$.

For the rest of the proofs we make the transformation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ m/n & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

which shows that $S(m)$ is the number of integer lattice points inside the triangle $A' = (0,0)$, $B' = (0,1)$, $C' = (n/2, m/2)$ plus the number of lattice points on $B'C'$, which is zero if $m \neq 2$.

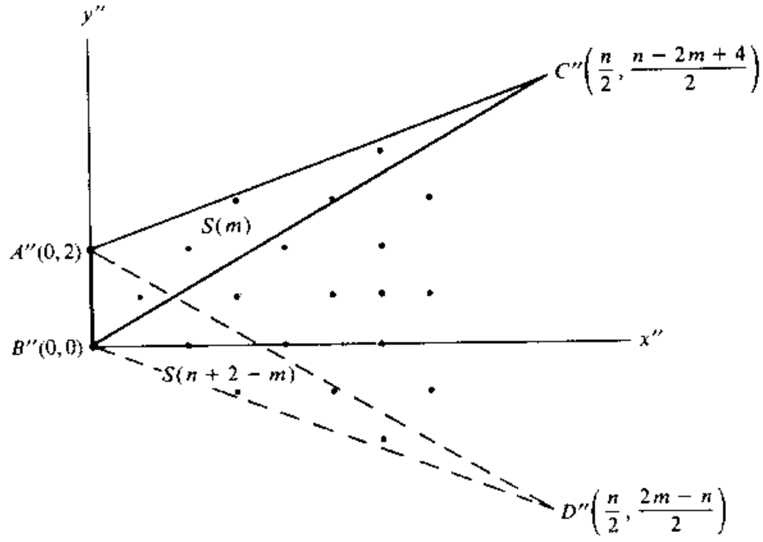

FIG. 1. The symmetry about $y'' = 1$ showing $S(m) = S(n+2-m)$ for $3 \leqslant m \leqslant n-1$.

To prove (ii), in case $m = 2$ equation (1) has $\left\lfloor \frac{n}{2} \right\rfloor$ solutions, and there are no points inside the triangle, so $S(2) = \left\lfloor \frac{n}{2} \right\rfloor$. It is easy to check $S(0) = S(1) = 0$, $S(3) = \left\lfloor \frac{n}{3} \right\rfloor$.

To prove (iii), we shall estimate the general values of $S(m)$ by counting the integer lattice points inside the triangle $A'B'C'$ that lie on parallel lines whose slopes are rational, close to m/n, in a Farey series of small order f. The uncertainty in this estimate will be the number of parallel lines crossing the triangle. If n is large, the triangle is long and thin and the relative uncertainty can be reduced by an appropriate choice of f.

The Farey series of order f consists of the fractions l/k with $0 \leqslant l \leqslant k \leqslant f$ and $\gcd(l,k) = 1$. If l/k is such a fraction (close to m/n), then either (i) $km - ln = 2s$ or (ii) $km - ln = 2s + 1$ for some (small) integer s.

LEMMA 1. *If $km - ln = 2s$ and $4 \leqslant m \leqslant n-2$, then $S(m)$ lies in the range*

$$\frac{n}{4} \pm \max(k-1, s-1, k-s-1).$$

Proof. Let λ_i be the line $ky' - lx' = i$. Consider the three cases (ia) $0 \leqslant s \leqslant k$, (ib) $k < s$ and (ic) $s < 0$. In case (ia), if $0 < i \leqslant s$, then λ_i intersects the triangle $A'B'C'$ on $A'B'$ where $x' = 0$ and on $A'C'$ where $x' = in/(km - ln) = in/2s$. Since $\gcd(l,k) = 1$ it follows that the integer solutions, for x', to the Diophantine equation λ_i differ by multiples of k and the line λ_i contains either $\lfloor in/2ks \rfloor$ or $\lfloor in/2ks \rfloor + 1$ integer lattice points inside the triangle $A'B'C'$. That is, the number of lattice points lies in the range $(in/2ks) \pm 1$.

Similarly, if $s < i < k$, the line λ_i meets the sides $A'B'$ and $B'C'$ and the number of lattice points on λ_i inside $A'B'C'$ lies in the range $((k-i)n/2k(k-s)) \pm 1$. However,

$$\sum_{i=1}^{s} \frac{in}{2ks} + \sum_{i=s+1}^{k-1} \frac{(k-i)n}{2k(k-s)} = \frac{n}{4}$$

so $S(m)$, the total number of lattice points inside $A'B'C'$, lies in the range $(n/4) \pm (k-1)$.

Similarly, in case (ib), $S(m)$ lies in the range $(n/4) \pm (s-1)$ and, in case (ic), $S(m)$ lies in the range $(n/4) \pm (k-s-1)$. □

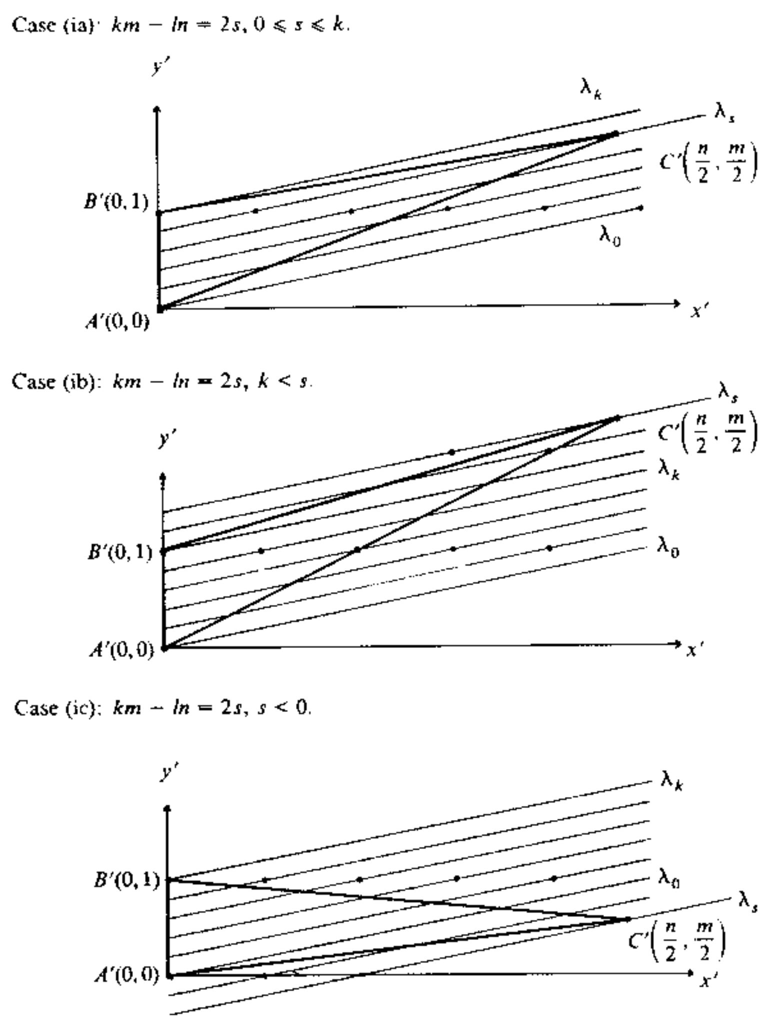

FIG. 2(i). The three cases when $km - ln$ is even.

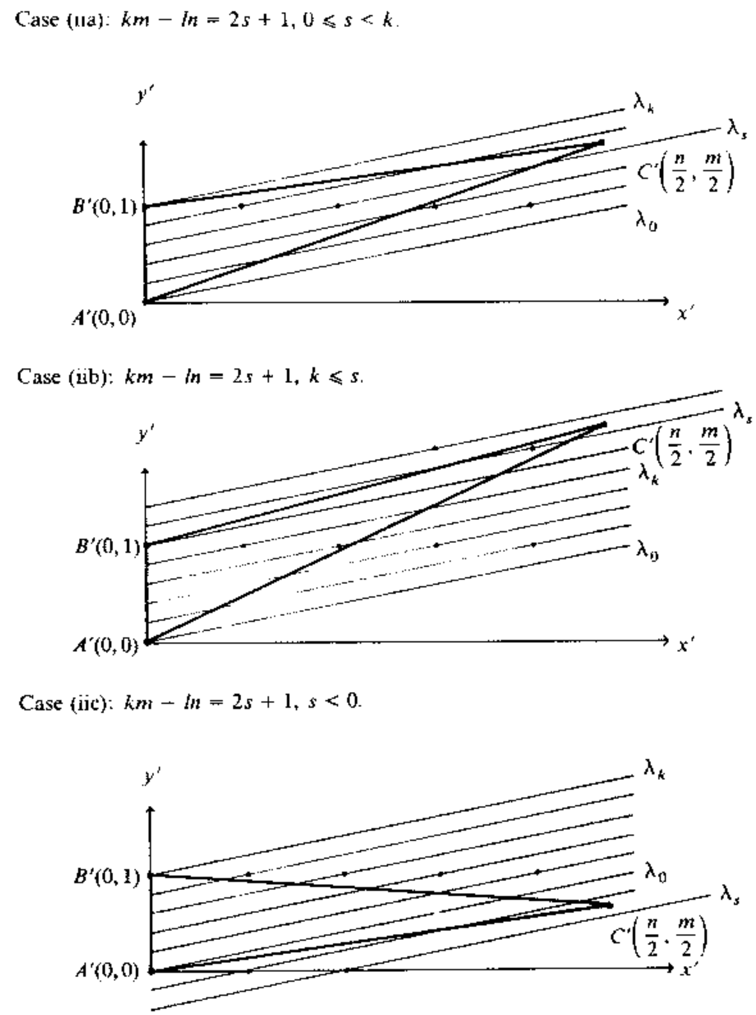

FIG. 2(ii). The three cases when $km - ln$ is odd.

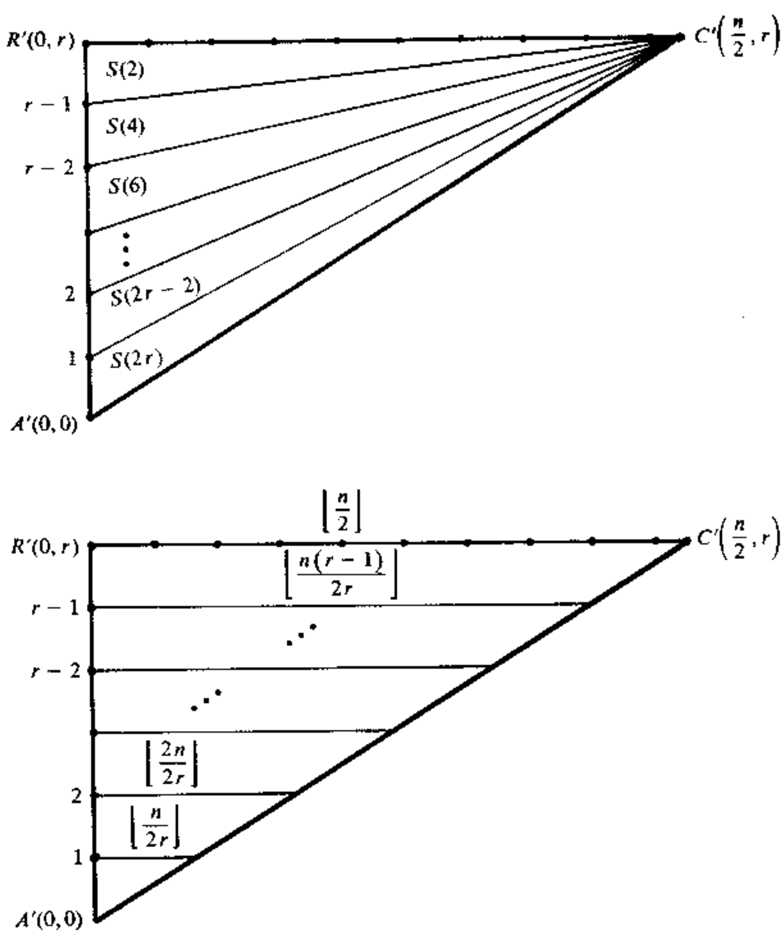

FIG. 3(i). Illustration of the equality $\sum_{i=1}^{r} S(2i) = \sum_{j=1}^{r} \left\lfloor \frac{n_j}{2r} \right\rfloor$.

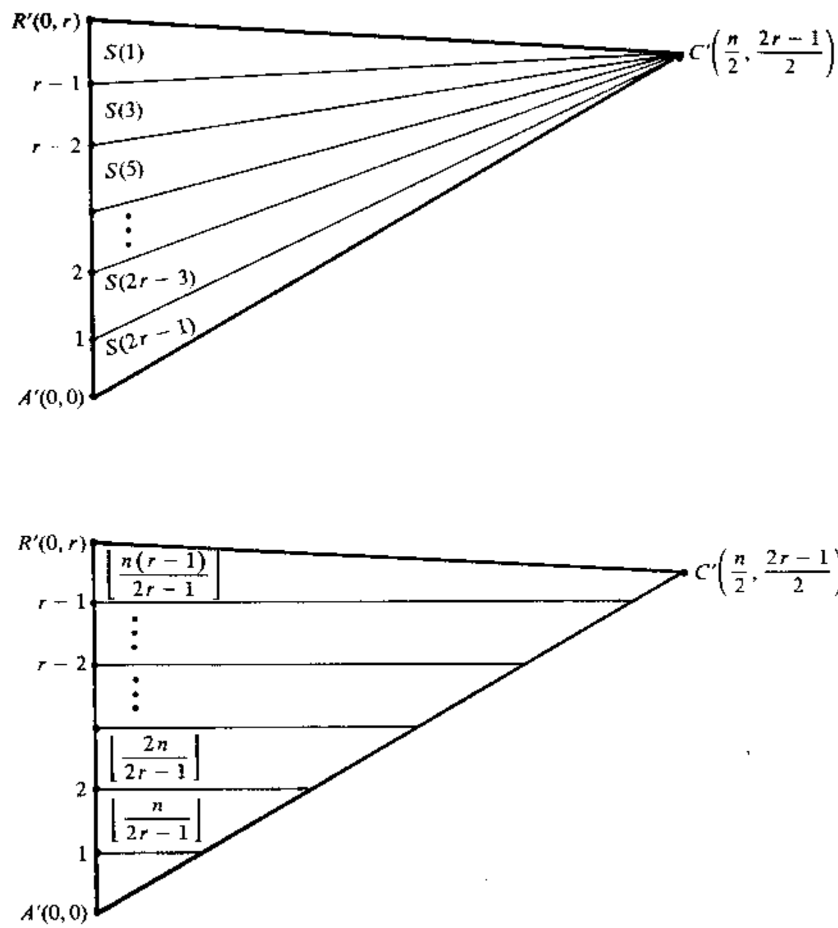

FIG. 3(ii). Illustration of the equality $\sum_{i=1}^{r} S(2i-1) = \sum_{j=1}^{r-1} \left\lfloor \frac{n_j}{2r-1} \right\rfloor$.

LEMMA 2. *If $km - ln = 2s + 1$ and $4 \leqslant m \leqslant n - 2$, then $S(m)$ lies in the range*

$$\frac{n}{4}\left(1 + \frac{1}{(2s+1)(2s-2k+1)}\right) \pm \max(k-1, s, k-s-1).$$

This lemma is proved in a similar way to Lemma 1 by considering the three cases (iia) $0 \leqslant s < k$, (iib) $k \leqslant s$ and (iic) $s < 0$.

In particular, if $m = (n+1)/2$, take $l/k = 1/2$ so that $km - ln = 1$ and $S((n+1)/2)$ lies in the range $(n/6) \pm 1$. Looking more closely at the line λ_1, we see that its integer points occur when x' is odd and $S((n+1)/2) = (n-1)/6$ if $n \equiv 1 \pmod 6$ and $S((n+1)/2) = (n+1)/6$ if $n \equiv 5 \pmod 6$. Hence $S((n+1)/2) = \lfloor (n+1)/6 \rfloor$. In a similar way, if $m = (n-1)/2$, take $l/k = 1/2$ so that $km - ln = -1$ and $S((n-1)/2)$ lies in the range $(3n/10) \pm 2$. Examining λ_0 and λ_1 in more detail for the separate cases $n \equiv 1, 3, 7, 9 \pmod{10}$, we see that $S((n-1)/2) = \lfloor 3n/10 \rfloor$.

LEMMA 3. *If there exists a Farey fraction l/k of order $f = \lfloor (n-150)/20 \rfloor$ such that $2(k-f) \leqslant km - ln \leqslant 2f$, then $\lfloor (n+1)/6 \rfloor \leqslant S(m) \leqslant \lfloor 3n/10 \rfloor$ for $4 \leqslant m \leqslant n-2$.*

Proof. In case (i) when $km - ln = 2s$, it follows from Lemma 1 that $S(m)$ lies in the range $(n/4) \pm (f-1)$. Now $(n/4) + f - 1 < 3n/10$ since $f < n/20$ and $(n/4) - f + 1 > (n+1)/6$ since $f < (n/12) - 1$. Hence $\lfloor (n+1)/6 \rfloor \leqslant S(m) \leqslant \lfloor 3n/10 \rfloor$.

In case (iia) when $km - ln = 2s + 1$ and $0 \leqslant s < k$, it follows from Lemma 2 that $S(m)$ lies in the range

$$\frac{n}{4}\left(1 - \frac{1}{(2s+1)(2k-2s-1)}\right) \pm (k-1).$$

Hence $S(m) < (n/4) + (f-1) < 3n/10$ since $f < n/20$. The minimum value of $(2s+1) \times (2k-2s-1)$ is $2k-1$ and occurs when $s = 0$ or $k-1$. Therefore, since $k \leqslant f < (n-29)/12$, it follows that

$$\begin{aligned} \frac{n}{4}\left(1 - \frac{1}{2k-1}\right) - k + 1 - \frac{n+1}{6} &= \frac{n(k-2)}{6(2k-1)} - \frac{6k-5}{6} \\ &= \frac{(k-2)}{6(2k-1)}\left(n - 12k - 8 - \frac{21}{k-2}\right) > 0 \end{aligned}$$

if $k > 2$. Hence, if $k > 2$,

$$S(m) \geqslant \frac{n}{4}\left(1 - \frac{1}{2k-1}\right) - k + 1 > \frac{n+1}{6} \geqslant \left\lfloor \frac{n+1}{6} \right\rfloor.$$

Now k cannot be 1 since $m \neq 1$. If $k = 2$, then $s = 0$ or 1 and so $m = (n+1)/2$ or $(n+3)/2$. By the symmetry,

$$S((n+3)/2) = S((n+1)/2) = \lfloor (n+1)/6 \rfloor.$$

In case (iib) when $km - ln = 2s + 1$ and $k \leqslant s \leqslant f - 1$, it follows from Lemma 2 that $S(m)$ lies in the range

$$\frac{n}{4}\left(1 + \frac{1}{(2s+1)(2s-2k+1)}\right) \pm s.$$

Hence $S(m) > (n/4) - s > (n+1)/6 \geqslant \lfloor (n+1)/6 \rfloor$, since $s < f < (n/12) - 1$. The minimum value of $(2s+1)(2s-2k+1)$ is $2s+1$ and occurs when $k = s$. Therefore, since $s < f < (n-150)/20$, it follows that

$$\frac{n}{4}\left(1 + \frac{1}{2s+1}\right) + s - \frac{3n}{10} = \frac{-n(s-2)}{10(2s+1)} + s$$

$$= -\frac{(s-2)}{10(2s+1)}\left(n - 20s - 50 - \frac{100}{s-2}\right) < 0$$

if $s > 2$. Hence, if $s > 2$,

$$S(m) \leqslant \frac{n}{4}\left(1 + \frac{1}{2s+1}\right) + s < \frac{3n}{10};$$

therefore $S(m) \leqslant \lfloor 3n/10 \rfloor$. Now s cannot be 1 since $m \neq 3$. If $s = 2$, then $k = 1$ or 2 and so $m = 5$ or $(n+5)/2$. Now, by Lemma 2, $S(5) \leqslant (4n/15) + 2 < 3n/10$, if $n > 60$. However, the conditions of Lemma 3 implicitly imply that $n > 170$, since $f \geqslant 1$. By the symmetry, $S((n+5)/2) = S((n-1)/2) = \lfloor 3n/10 \rfloor$.

The proof of the case (iic), when $km - ln = 2s + 1$ and $s < 0$, is similar to the case (iib). □

LEMMA 4. *If $n > 530$ and $f = \lfloor (n-150)/20 \rfloor$, then*

$$\frac{ln + 2f}{2k} > \frac{l'n + 2(k'-f)}{2k'},$$

where l/k and l'/k' are successive terms in the Farey series of order f.

Proof. Successive terms in a Farey series satisfy $l'k - lk' = 1$ and $k + k' \geqslant f + 1$. (See G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, Ch. III.) In the range $1 \leqslant k \leqslant f$, $1 \leqslant k' \leqslant f$ and $k + k' \geqslant f + 1$, the minimum value of $2f(k+k') - 2kk'$ is $(3f^2 + 2f - 1)/2$. If $f = \lfloor (n-150)/20 \rfloor$ and $n > 530$, then $(3f^2 + 2f - 1)/2 > n$. Hence, if $n > 530$,

$$2f(k+k') - 2kk' > n = (l'k - lk')n.$$

Dividing by $2kk'$ we obtain the desired result. □

Lemma 4 implies that the conditions of Lemma 3 are satisfied when $n > 530$ and so the result is true in these cases. (This value of 530 could be reduced to about 300 by considering each of the low values of $(2s+1)(2s-2k+1)$ separately in Lemma 3 and raising the value of f appropriately.)

For $n < 530$, the result was checked using a programmable calculator.