E 2927* [1982, 130]. *Proposed by Clark Kimberling, University of Evansville.*

Define sequences $\{a_n\}, \{b_n\}, \{c_n\}$ inductively as follows: $a_1 = 1$, $b_1 = 2$, $c_1 = 4$, and take

$a_n =$ least positive integer not among $a_1, \ldots, a_{n-1}, b_1, \ldots, b_{n-1}, c_1, \ldots, c_{n-1}$

$b_n =$ least positive integer not among $a_1, \ldots, a_{n-1}, a_n, b_1, \ldots, b_{n-1}, c_1, \ldots, c_{n-1}$

$c_n = 2b_n + n - a_n$.

Prove or disprove that $0 < n(1+\sqrt{3}) - b_n < 2$ for all n.

Solution by William J. Gilbert, University of Waterloo, Canada. We will prove that $\alpha < n(1+\sqrt{3}) - b_n < \beta$ for all n, where $\alpha = (9 - 5\sqrt{3})/3 > 0.113$ and $\beta = (12 - 4\sqrt{3})/3 < 1.691$.

First, prove by induction that $1 \le b_n - a_n \le 2$, $1 \le a_{n+1} - b_n \le 2$ and $2 \le c_{n+1} - c_n \le 6$ for all n, using the equation

$$(*) \qquad c_{n+1} - c_n = 2(b_{n+1} - a_{n+1}) + (a_{n+1} - b_n) - (b_n - a_n) + 1.$$

Secondly, let $\gamma_n = n(1+\sqrt{3}) - b_n$. We will prove by induction that $\alpha < \gamma_n < \beta$. It is true that $\alpha < \gamma_1 < \beta$. Suppose $\alpha < \gamma_n < \beta$ for all $n < m$. By the previous result, $b_m, a_m, b_{m-1}, a_{m-1}, b_{m-2}, a_{m-2}$ cannot be six consecutive integers. Let c_n be the first integer missing in this decreasing sequence. Then $n < m$ and either (a) $c_n = a_{m-j} - 1$ for $j = 0$ or 1 or (b) $c_n = b_{m-j} - 1$ for $j = 0$, 1 or 2. Hence $a_1, \ldots, a_m, b_1, \ldots, b_m, c_1, \ldots, c_n$ are the first b_m integers and either (a) $2m + n = b_m = c_n + 2 + 2j$ or (b) $2m + n = b_m = c_n + 1 + 2j$. Now $c_n = b_n + n + (b_n - a_n)$ so, in all cases, $2m + n = b_m = b_n + n + k$ where $2 \le k \le 7$. If $j = 2$, then $c_{n+1} - c_n = 6$ and from (*) it follows that $b_n - a_n = 1$. Hence k cannot be 7. Therefore $2m = b_n + k$ for $2 \le k \le 6$.

Now $\gamma_m = (1+\sqrt{3})m - b_m = (1+\sqrt{3})(b_n + k)/2 - (b_n + n + k) = (k - \gamma_n)(\sqrt{3} - 1)/2$. Using the induction hypothesis, if $2 \le k \le 6$, then $\gamma_m > (2 - \beta)(\sqrt{3} - 1)/2 = \alpha$ and, if $2 \le k \le 4$, then $\gamma_m < (4 - \alpha)(\sqrt{3} - 1)/2 < \beta$. If $k = 6$, it follows from (*) that $b_{n+1} - a_{n+1} = a_{n+1} - b_n = 2$ and so $b_{n+1} - b_n = 4$. Now $\gamma_{n+1} - \gamma_n = 1 + \sqrt{3} - (b_{n+1} - b_n) = \sqrt{3} - 3$. However $n + 1 < m$ so, using the induction hypothesis, $\gamma_n = \gamma_{n+1} + 3 - \sqrt{3} > \alpha + 3 - \sqrt{3}$. Therefore, if $k = 6$, $\gamma_m < (6 - \alpha - 3 + \sqrt{3})(\sqrt{3} - 1)/2 = \beta$. Similarly, if $k = 5$, it follows from (*) that $b_{n+1} - b_n = 3$ or 4. Hence $\gamma_{n+1} - \gamma_n \le \sqrt{3} - 2$ and $\gamma_n > \alpha + 2 - \sqrt{3}$. Therefore $\gamma_m < (5 - \alpha - 2 + \sqrt{3}) \cdot (\sqrt{3} - 1)/2 = \beta$ which completes the induction.

Edmund Butler, Tim Keller and Paul Pudaite also established the stronger inequalities on $x_n = n(1+\sqrt{3}) - b_n$ given in the above solution.

It is claimed that numerical evidence suggests these bounds are the best possible. For example, at $n = 17{,}135$, $x_n = 1.690588$ while $x_n = 0.113253$ at $n = 23{,}407$.

Also solved by D. M. Bloom, R. Breusch, E. Butler, D. Finkel, Y. Hong, T. Keller, L. E. Mattics, and P. Pudaite.