Linear Orderings of Finite Sets in R^n

6394 [1982, 502]. *Proposed by Jan Mycielski and Andrzej Ehrenfeucht, University of Colorado, Boulder.*

Let n and $m \geqslant n+1$ be given. Let $X \subseteq R^n$ with $|X| = m$ be a set in general position, i.e., if $Y \subseteq X$ and $|Y| \leqslant n+1$, then Y spans a $(|Y|-1)$-dimensional hyperplane and no two such hyperplanes are parallel to each other if, of the corresponding two Y's, neither is a subset of the other. Let $f_n(m)$ be the number of linear orderings of X which can be obtained by a perpendicular projection of X into any directed line which is in general position relative to X.

(a) Prove that $f_2(m) = 2\binom{m}{2}$ and $f_3(m) = 6\binom{m}{4} + 4\binom{m}{3} + 2$.

(b) Is $f_n(m)$ well defined (i.e., the same for all X in general position) for $n \geqslant 4$, and, if so, can it be evaluated?

Solution by W. J. Gilbert, University of Waterloo, Canada, and A. Mandel, University of Sao Paulo, Brazil. We give a direct proof of (a) and examples to show that $f_4(6)$ is not well defined.

The projections of X onto parallel lines directed in the same way give the same ordering so, for each parallel class, we choose the representation through the origin parameterized as $l_c(\lambda) = \{\lambda c | \lambda \in R\}$, where $c \in S^{n-1}$, the unit $(n-1)$-sphere in R^n. Let $X = \{x^1, \ldots, x^m\}$. The projection of the point x^i onto the line l_c has parameter $\lambda = x^i \cdot c$; hence x^i precedes x^j in the ordering on l_c if and only if $(x^i - x^j) \cdot c < 0$. For $1 \leqslant i < j \leqslant m$, let $h(i, j)$ be the $(n-1)$-dimensional

hyperplane $(x^i - x^j) \cdot x = 0$ which passes through the origin and is perpendicular to the line joining x^i to x^j. The side of $h(i, j)$ where c lies determines whether x^i precedes or follows x^j in the ordering on l_c. Therefore the number of distinct orderings of X equals the number of full dimensional regions that the collection of hyperplanes $\{h(i, j)\}$ divides R^n, or equivalently, divides S^{n-1}.

For $n = 2$, the general position of X just implies that all the $h(i, j)$ are distinct. There are $\binom{m}{2}$ of these, each intersecting S^1 in a pair of antipodal points. Hence S^1 is divided into $2\binom{m}{2}$ arcs and $f_2(m) = 2\binom{m}{2}$.

For $n = 3$, the general position of X implies that the planes $h(i, j)$ are distinct and the only triples of $h(i, j)$ that have a one dimensional intersection are those of the form $\{h(i, j), h(i, k), h(j, k)\}$, where $1 \leqslant i < j < k \leqslant m$. Look at the complex defined by the intersection of the planes $h(i, j)$ on the sphere S^2. Let v_{2r} be the number of vertices of degree $2r$; each such vertex and its antipode lie on the one dimensional intersection of r of the planes $h(i, j)$. Hence v_6 is twice the number of triples $\{i, j, k\}$ with $1 \leqslant i < j < k \leqslant m$ and $v_6 = 2\binom{m}{3}$. Now v_4 is twice the number of pairs $\{\{i, j\}, \{k, l\}\}$ with i, j, k, l distinct and $v_4 = 6\binom{m}{4}$. The number of vertices of the complex is $v_4 + v_6$ and the number of edges, e, is half the total degree, so $e = (4v_4 + 6v_6)/2$. By Euler's formula on the sphere, the number of faces is

$$f_3(m) = e - (v_4 + v_6) + 2 = v_4 + 2v_6 + 2 = 6\binom{m}{4} + 4\binom{m}{3} + 2.$$

To show that $f_4(6)$ is not well defined consider, for example, the following matrices, identified with their column sets:

$$X_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 5 \end{pmatrix} \quad \text{and} \quad X_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 12 \\ 0 & 0 & 1 & 0 & 0 & 8 \\ 0 & 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & -4 \end{pmatrix}.$$

Both X_1 and X_2 are in general position, but the above methods and Euler's formula on S^3 can be used to show that the number of orderings for X_1 is 480 while the number for X_2 is 472.

Part (a) was also solved by Pei Yuan Wu (Republic of China) and the proposers.