A Constrained Equality

5935 [1973, 1067]. *Proposed by K. Selucký, Brno, Czechoslovakia*

Suppose

$$\frac{1}{x_1} + \frac{1}{s - x_2} + \frac{1}{s - x_3} + \frac{1}{x_4} = \frac{1}{s - x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{s - x_4},$$

where $s > x_1 \geqq x_2 \geqq x_3 \geqq x_4 \geqq \frac{1}{2}s$. Prove $x_1 = x_2$ and $x_3 = x_4$.

I. *Solution by R. O. Davies, The University of Leicester, England.* Rewriting the equation as

$$\frac{2x_2 - s}{(s - x_2)x_2} + \frac{2x_3 - s}{(s - x_3)x_3} = \frac{2x_1 - s}{(s - x_1)x_1} + \frac{2x_4 - s}{(s - x_4)x_4},$$

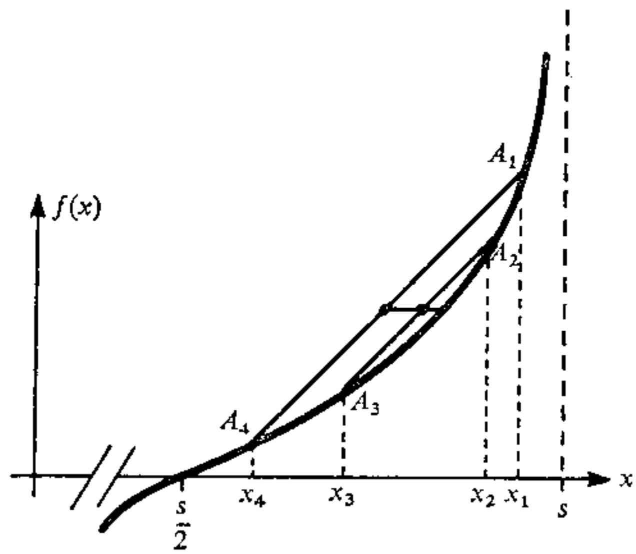
and observing that $(2x-s)/(s-x)\cdot x$ is strictly increasing (from 0 to ∞) in the range $s/2 < x < s$, we see that counterexamples are easily found. It is more difficult to find integer ones. I decided to try putting $x_2 = x_3$ and $x_4 = (3s/4) - t$, $x_1 = (3s/4) + t$ and by making judicious use of the known solution of $x^2 + y^2 = z^2$, I was able to discover the following:

$$s = 104,\ x_1 = 91,\quad x_2 = x_3 = 84,\ x_4 = 65.$$

II. *Solution by David Hertzig, University of Miami.* The change of variables $x_i = \frac{1}{2}(1 + \tan(\frac{1}{2}\theta_i))s$ converts the problem to:

Suppose $\tan\theta_1 - \tan\theta_2 = \tan\theta_3 - \tan\theta_4$, where $\frac{1}{2}\pi > \theta_1 \geqq \theta_2 \geqq \theta_3 \geqq \theta_4 \geqq 0$. Prove $\theta_1 = \theta_2$ and $\theta_3 = \theta_4$. Further, letting $y_i = \tan\theta_i$ gives $y_1 - y_2 = y_3 - y_4$, where $y_1 \geqq y_2 \geqq y_3 \geqq y_4 \geqq 0$. In this form, it is manifestly false that necessarily $y_1 = y_2$ and $y_3 = y_4$. For example, take $y_1 = 12/5$, $y_2 = 15/8$, $y_3 = 3/4$, $y_4 = 9/40$ corresponding to $x_1 = \frac{5}{6}s$, $x_2 = \frac{4}{5}s$, $x_3 = \frac{2}{3}s$, $x_4 = \frac{5}{9}s$.

III. *Solution by W. J. Gilbert, University of Waterloo, Ontario.* The problem as stated is clearly false. Perhaps there is some information missing. (If we impose the condition $x_1 + x_4 \geqq x_2 + x_3$, then it will follow, from the concavity, that $x_1 = x_2$ and $x_3 = x_4$.)



Let $f(x) = 1/(s-x) - 1/x$, so that the equality can be written as $f(x_2) + f(x_3) = f(x_1) + f(x_4)$. Let A_i be the point $(x_i, f(x_i))$. Then the equality holds when the midpoint of A_1A_4 is on the same level as the midpoint of A_2A_3. The graph of $f(x)$ is concave upwards when $s/2 < x < s$ because $f''(x) = 2/(s-x)^3 - 2/x^3 > 0$ if $0 < s - x < x$. Hence, for any choices of s, x_4, x_3 and x_2, such that $s > x_2 \geqq x_3 > x_4 \geqq s/2$ the equality can be solved for x_1 to give a counterexample. For example, if $s = 1$, $x_4 = \frac{1}{2}$, $x_3 = x_2 = \frac{2}{3}$, then $x_1 = (1 + \sqrt{13})/6$.

Also solved by Bennett College Team, J. C. Binz (Switzerland), James Boone & Mary King, Robert Breusch (New Zealand), L. E. Clarke (England), George Crofts, E. de Jonge (Netherlands), G. J. Ford, D. P. Giesy, F. Gobel (Netherlands), M. G. Greening (Australia), A. C. Hindmarsh, R. A. Jacobson, P. T. Joslin, Woon-Chung Lam & Radha G. Nath, O. P. Lossers (Netherlands), D. E. Mackenzie (Australia), L. E. Mattics, S. C. Otermat, J. P. Robertson, Saint Olaf College Students, J. W. Shaw, Jr., E. D. Shirley, Michael Skalsky, and Southern University Primer for Research Group. There were seven alleged proofs of the stated result.