SOME PROBLEMS IN ALGEBRAIC TOPOLOGY ON LUSTERNIK-SCHNIRELMANN CATEGORIES AND COCATEGORIES

by

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ABSTRACT

In this thesis we are concerned with certain numerical invariants of homotopy type akin to the Lusternik-Schnirelmann category and cocategory.

In a series of papers I. Berstein, T. Ganea, and P. J. Hilton developed the concepts of the category and weak category of a topological space. They also considered the related concepts of conilpotency and cup product length of a space and the weak category of a map. Later T. Ganea gave another definition of category and weak category (which we will write as G-cat and G-wcat) in terms of fibrations and cofibrations and hence this dualizes easily in the sense of Eckmann-Hilton.

We find the relationships between these invariants and then find various examples of spaces which show that the invariants are all different except cat and G-cat. The results are contained in the following theorem. The map $e:B \longrightarrow \Omega\Sigma B$ is the natural embedding. All the invariants are normalized so as to take the value 0 on contractible spaces.

<u>Theorem.</u> Let B have the homotopy type of a simply connected CW-complex, then

cat B=G-cat B $\geq$ G-wcat B $\geq$ wcat B $\geq$ wcat e $\geq$ conil B $\geq$ $\cup$-long B

and furthermore all the inequalities can occur.

All the examples are spaces of the form $B = S^q \cup_\alpha e^n$ where $\alpha \in \pi_{n-1}(S^q)$. When B is of this form, we obtain conditions for the category and and weak categories of B to be less than or equal to one in terms of Hopf invariants of α. We use these conditions to prove the examples.

We then prove the dual theorem concerning the relationships between the invariants cocategory, weak cocategory, nilpotency and Whitehead product length.

<u>Theorem.</u> Let A be a countable CW-complex, then

cocat A $\geq$ wcocat A $\geq$ nil A $\geq$ W-long A

and furthermore all the inequalities can occur.

The proof is not dual to the first theorem, though the examples we use to show that the inequalities can exist are all spaces with two non-zero homotopy groups.

The most interesting of these examples is the space A with 2 non-zero homotopy groups, $\mathbb{Z}$ in dimension 2 and $\mathbb{Z}_4$ in dimension 7 and with k-invariant $u^4 \in H^8(\mathbb{Z},2;\mathbb{Z}_4)$. This space is not an H-space, but has weak cocategory 1. The condition wcocat $A \leqslant 1$ is equivalent to the fact that $d \simeq 0$ in the fibration $D \xrightarrow{d} A \xrightarrow{e} \Omega\Sigma A$. In order to show that wcocat $A = 1$ we have to calculate the cohomology ring of $\Omega\Sigma K(\mathbb{Z},2)$. The method we use to do this is the same as that used to calculate the cohomology ring of ΩS^{n+1} using James' reduced product construction.

Finally we show that for the above space A the fibration

$$\Omega A \xrightarrow{g} A^S \xrightarrow{f} A$$

has a retraction ρ such that $\rho \circ g \simeq 1$ even though A is not an H-space.

CONTENTS

0. INTRODUCTION AND SUMMARY

In their work in Differential Geometry and the Calculus of Variations, Lusternik and Schnirelmann [24] introduced the notion of the category of a space. This has interest in Topology because category turns out to be a numerical invariant of homotopy type (as noted by Borsuk [7]) which is quite different from the usual topological invariants.

There are various definitions of category given in Chapters 1 and 4 but they all give the same value for most spaces. In this thesis all the definitions will be normalized so that a contractible space has category 0.

There is a connection between the category of a space and the homology invariants which comes from the fact that the cup product length of a space (i.e. the largest number of cohomology classes with a non-vanishing cup product) is a lower bound for the category. If the space is a CW-complex then the number of positive dimensional cells is an upper bound for the category.

In a series of papers Berstein, Ganea, Hilton and Peterson [3], [4], [5], [12], [14] discover various numerical invariants whose values lie sandwiched between the values of the category and the cup product length. We investigate the relationships between the various invariants and by examples show that, except for the various definitions of category, all the other invariants are different. The results are collected together in the following theorem.

Theorem 4.5. Let B have the homotopy type of a simply connected countable CW-complex, then

$$\text{cat } B = G\text{-cat } B \geqslant G\text{-wcat } B \geqslant \text{wcat } B \geqslant \text{wcat } e \geqslant \text{conil } B \geqslant \cup\text{-long } B$$

and furthermore all the inequalities can occur.

All the examples will be spaces of the form $S^q \cup_\alpha e^n$ where $\alpha \in \pi_{n-1}(S^q)$ and we will use Toda's notation [33] for the elements of the homotopy groups of spheres.

The spaces with category 1 are precisely the non-contractible coH-spaces (i.e. spaces with a comultiplication). The coH-spaces are dual in the sense of Eckmann-Hilton to the H-spaces. Hence it is natural to consider the dual concepts to category and the other related invariants.

Not all the definitions of categories and weak categories dualize straightforwardly. But the main point of Ganea's definitions is that they do dualize easily and hence we will use these duals to define cocategory and weak cocategory. Then the spaces of cocategory 1 are precisely the non-contractible H-spaces.

An upper bound for the cocategory of a simply connected CW-complex is the number of non-zero homotopy groups.

The following theorem is dual to Theorem 4.5 since, in Eckmann-Hilton duality, nilpotency is dual to conilpotency and the Whitehead product length is dual to the cup product length.

<u>Theorem 6.5.</u> Let A be a countable CW-complex, then

$$\text{cocat } A \geqslant \text{wcocat } A \geqslant \text{nil } A \geqslant \text{W-long } A$$

and furthermore all the inequalities can occur.

The examples we use to prove that the inequalities can occur are all spaces with two non-zero homotopy groups, but the methods are not dual to those of Theorem 4.5.

T. Ganea has pointed out that an example of a space with weak cocategory 1 but which is not an H-space can be used to answer, in the negative, the following question of I. M. James. If A^S is the space of free loops on A and in the following natural fibration

$$\Omega A \xrightarrow{g} A^S \xrightarrow{f} A$$

there is a retraction ρ such that $\rho \circ g \simeq 1$ does it follow that A is an H-space?

The thesis is divided into eight chapters. In Chapter 1 we give the older definitions of category and the definitions of the related invariants. We mention the known examples of spaces which distinguish the four invariants cat B, wcat B, conil B and $\cup$-long B.

In Chapter 2 we restrict our attention to spaces B of the form $S^q \cup_\alpha e^n$ and find a condition for wcat e $\leqslant 1$ in terms of a Hopf invariant of α. We use this to find examples of spaces which distinguish wcat e from wcat B and conil B.

Following Svarc [32] we introduce in Chapter 3 the 'sum' of fibrations over the same base and the 'product' of fibrations over different bases.

The main result we need later is Theorem 3.4 when applied to the standard fibration $\partial\mathcal{B}: \Omega B \longrightarrow PB \longrightarrow B$. Of course this result could be obtained directly without using 'sums' and 'products' of fibrations (see [15]) but the result and certainly the proof is clearer when stated in general terms.

In Chapter 4 we give Ganea's definition of category in terms of 'sums' of the standard fibration $\partial\mathcal{B}$. We then prove the relationships between this definition and the older ones.

In Chapter 5 we prove that G-wcat is different from both cat and wcat. Again we use spaces of the form $B = S^q \cup_\alpha e^n$ and find a condition for G-wcat $B \leq 1$ in terms of a composite Hopf invariant of α.

We introduce the dual invariants in Chapter 6 and find the relationships between them. We give examples of spaces which distinguish nil from wcocat and W-long.

In order to find an example to distinguish cocat from wcocat we will need to know the cohomology ring structure of $\Omega\Sigma K(\mathbb{Z},2) = \Omega\Sigma\mathbb{C}P$. We develope this in Chapter 7 and show that the same method can be used to calculate the cohomology ring structure of ΩS^{n+1} from the reduced product complex S^n_∞.

The space we find in Example 8.5, which has weak cocategory 1 but which is not an H-space, has 2 non-zero homotopy groups, $\mathbb{Z}$ in dimension 2 and $\mathbb{Z}_4$ in dimension 7 and the k-invariant is $u^4 \in H^8(\mathbb{Z},2;\mathbb{Z}_4)$. Finally we show that this space can be used to answer a question of I. M. James.

The contents of Chapters 2, 4 and 5 and a particular case of Theorem 3.4 will be published in [15].

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1. DEFINITIONS

All the spaces we will consider have the homotopy type of countable connected CW-complexes and have a base point denoted by *. All the maps will preserve base points. The constant map is denoted by 0 and the identity map by 1. We will not usually distinguish between a map and its homotopy class. Denote the set of homotopy classes of maps from X to Y by $[X;Y]$.

Let T_1^{k+1} be the subset of B^{k+1} consisting of points with at least one coordinate equal to *. Let $j: T_1^{k+1} \longrightarrow B^{k+1}$ be the inclusion map and let $B^{(k+1)}$ be the quotient space B^{k+1}/T_1^{k+1} with identification map $q: B^{k+1} \longrightarrow B^{(k+1)}$. Then $B^{(k+1)}$ is the (k+1)-fold smash product of B. Let $\triangle : B \longrightarrow B^{k+1}$ be the diagonal map.

We now recall the definitions of some of the numerical invariants akin to the Lusternik-Schnirelmann category. All the invariants will be normalized so as to take the value 0 on contractible spaces. (This means, for example, that we subtract 1 from the value of the classical Lusternik-Schnirelmann category.)

Definition 1.1. The Lusternik-Schnirelmann category of a space B, L-S cat B, is the least interger $k \geqslant 0$ with the property that B may be covered by (k+1) open subsets which are contractible in B; if no such integer exists L-S cat B $= \infty$.

Definition 1.2. The category of a space B, cat B, is the least integer $k \geqslant 0$ for which there exists a map $\varphi : B \longrightarrow T_1^{k+1}$ such that $j \circ \varphi \simeq \Delta : B \longrightarrow B^{k+1}$; if no such integer exists cat B $= \infty$.

The first definition is in the form given by Fox [11] when he modified that of Lusternik and Schnirelmann by using open subsets instead of closed subsets. The second definition was given by G. W. Whitehead [35] who observed that, for a certain class of spaces including all CW-complexes, it is equivalent to Definition 1.1.

The spaces with category 0 are the contractible spaces and the spaces with category 1 are the non-contractible coH-spaces. The obvious example of coH-spaces are the suspensions but there exist spaces of category 1 which are not suspensions [5; p. 444].

Since any 0-cell of a CW-complex in non-degenerate in the sense of Puppe [29] the following proposition follows from Definition 1.1.

Proposition 1.3. If B is a countable connected CW-complex with (k+1) cells then cat B $\leqslant$ k.

Whitehead's definition of category suggests a slightly weaker invariant called the weak category. It is also possible to give a definition of the category and weak category of a map so that cat B = cat 1_B where $1_B : B \longrightarrow B$ is the identity map. We will only be interested in the weak category of the map $e : B \longrightarrow \Omega\Sigma B$, which is the natural embedding of B into the loops on the suspension of B.

Definition 1.4. The weak category of a space B, wcat B, is the least integer $k \geqslant 0$ for which $q \circ \Delta \simeq 0 : B \longrightarrow B^{(k+1)}$; if no such integer exists wcat B $= \infty$.

Definition 1.5. The weak category of the map $e : B \longrightarrow \Omega\Sigma B$, wcat e, is the least integer $k \geqslant 0$ for which $q \circ \Delta \circ e \simeq 0 : B \longrightarrow (\Omega\Sigma B)^{(k+1)}$; if no such integer exists wcat e $= \infty$.

Finally there are two other invariants, the conilpotency and the cup product length, which are related to the category.

Let ψ_{n+1} be the co-commutator map of weight $n+1$ with respect to the comultiplication derived from the suspension ΣB of the space B [3; Definition 1.4].

Definition 1.6. The conilpotency of the suspension of B, conil B, is the least integer $k \geqslant 0$ for which $\psi_{k+1} \simeq 0$; if no such integer exists conil B $= \infty$.

Definition 1.7. The cup product length, $\cup$-long B, is the length of the longest non-trivial cup product of positive dimensional elements over any commutative ring.

It is clear from the definitions and from Theorems 4.4 and 5.2 of [14] that the following relationships hold between the invariants:

(1.8) cat B $\geqslant$ wcat B $\geqslant$ wcat e $\geqslant$ conil B $\geqslant$ $\cup$-long B.

The following results due to Berstein and Ganea show that if the space B is sufficiently connected then some of the invariants are the same.

<u>Proposition 1.9.</u> [4; Theorems 3 and 4] Let B be an (n-1) connected CW-complex.

(i) If the dimension of $B \leqslant (k+2)n-2$ and wcat $B \leqslant k$ then also cat $B \leqslant k$.

(ii) If the dimension of $B \leqslant 2(k+1)n-2$ and conil $B \leqslant k$ then also wcat $B \leqslant k$.

The first part of this proposition yields the best possible result as the following example shows.

<u>Example 1.10.</u> [5; p. 450] Let $B = (S^q \vee S^q) \cup_\alpha e^{3q-1}$ where the attaching map α is in the class of the triple Whitehead product $[\iota_1, [\iota_1, \iota_2]]$, ι_1 and ι_2 being the left and right embeddings $S^q \longrightarrow S^q \vee S^q$. Then cat B = 2 and wcat B = 1.

<u>Example 1.11.</u> [14; Theorem 6.1] Let $B = S^2 \cup_\alpha e^8$ where α is the generator of order 2 in $\pi_7(S^2)$. Then wcat B = 2 and conil B = 1.

<u>Example 1.12.</u> [3; p. 340] Let $B = S^2 \cup_\alpha e^5$ where α is the generator of order 2 in $\pi_4(S^2)$. Then conil B = 2 and $\cup$-long B = 1.

In the next chapter we will produce examples of strict inequalities between the invariant wcat e and the invariants wcat B and conil B.

2. WEAK CATEGORY OF THE MAP e

In this section we will obtain a criterion for wcat e $\leqslant 1$ when B is of the form $S^q \cup_\alpha e^n$. We then use this to find examples of spaces which distinguish wcat e from wcat B and conil B.

We recall the following conditions for cat B $\leqslant 1$ and wcat B $\leqslant 1$. The delicate Hopf invariant $\mathcal{H}: \pi_{n-1}(S^q) \longrightarrow \pi_n(S^q \times S^q, S^q \vee S^q)$ and the crude Hopf invariant $\overline{H}: \pi_{n-1}(S^q) \longrightarrow \pi_n(S^q \wedge S^q)$ are defined by Berstein and Hilton in [5; (2.11)]. See Proposition 5.2 for the connection between these Hopf invariants and the other Hopf invariants we define in Chapter 5.

<u>Theorem 2.1.</u> [5; Theorem 3.20] Let $B = S^q \cup_\alpha e^n$. Then (i) cat B $\leqslant 1$ if and only if $\mathcal{H}(\alpha) = 0$

(ii) wcat B $\leqslant 1$ if and only if $\overline{H}(\alpha) = 0$.

The arguments used in the proof of part(ii) of the above theorem may be adapted to prove the following theorem.

<u>Theorem 2.2.</u> Let $B = S^q \cup_\alpha e^n$. Then wcat e $\leqslant 1$ if and only if $(e \wedge e)_* \overline{H}(\alpha) = 0 \in \pi_n(\Omega\Sigma S^q \wedge \Omega\Sigma S^q)$.

Here $(e \wedge e)$ is the map from the smash product $S^q \wedge S^q$ which is e on each factor.

For completeness, we add the following result mentioned by Ganea, Hilton and Peterson.

<u>Theorem 2.3.</u> [14; p.140] Let $B = S^q \cup_\alpha e^n$. Then conil $B \leqslant 1$ if and only if $\Sigma \overline{H}(\alpha) = 0$.

The following lemma gives a simpler expression for the space $\Omega\Sigma S^q \wedge \Omega\Sigma S^q$.

<u>Lemma 2.4.</u> For q even, $\Omega\Sigma S^q \wedge \Omega\Sigma S^q$ has the same $(5q-2)$-homotopy type as the cell complex $T = S^{2q} \cup_\gamma e^{4q} \vee S^{3q} \vee S^{3q} \vee S^{4q} \vee S^{4q}$ where $\gamma = 2\,[\iota_{2q}, \iota_{2q}] \in \pi_{4q-1}(S^{2q})$.

<u>Proof.</u> The space $\Omega\Sigma S^q$ is homotopic to S^q_∞, the reduced product complex of James [21], which has a cellular decomposition

$$S^q_\infty = S^q \cup_\xi e^{2q} \cup_\zeta e^{3q} \quad \ldots \quad .$$

Now Milnor [25; Theorem 5] proves that

$$\Sigma\Omega\Sigma S^q \simeq S^{q+1} \vee S^{2q+1} \vee S^{3q+1} \vee \ldots$$

and hence it follows that the suspensions of the attaching maps ξ, ζ, etc. in S^q_∞ are trivial.

In the complex $S^q_\infty \wedge S^q_\infty$ there are the following cells of dimension less than 5q. There is one 0-cell and one 2q-cell. There are two 3q-cells attached by the maps $\Sigma^q \xi \simeq 0$ and two 4q-cells attached by the maps $\Sigma^q \zeta \simeq 0$. The remaining cell is a 4q-cell with an attaching map which we shall call $\beta \in \pi_{4q-1}(S^{2q} \vee S^{3q} \vee S^{3q})$. By the direct sum decomposition in [18] we can consider β an element of

$$\pi_{4q-1}(S^{2q}) \oplus \pi_{4q-1}(S^{3q}) \oplus \pi_{4q-1}(S^{3q}).$$

Now both components of β in $\pi_{4q-1}(S^{3q})$ factor through $\Sigma^{2q}\xi \simeq 0$. Let γ be the component of β in $\pi_{4q-1}(S^{2q})$. But

$$\begin{aligned}\tilde{H}^*(S^{2q} \cup_\gamma e^{4q} \vee S^{3q} \vee S^{3q}) &\approx \tilde{H}^*(S^q \cup_\xi e^{2q} \wedge S^q \cup_\xi e^{2q}) \\ &\approx \tilde{H}^*(S^q \cup_\xi e^{2q}) \otimes \tilde{H}^*(S^q \cup_\xi e^{2q})\end{aligned}$$

since there is no torsion. Let the generators on the left hand side corresponding to the 2q and the 4q-cells be u and v. Let the generators of $\tilde{H}^*(S^q \cup_\xi e^{2q})$ be w_1 and w_2 of dimensions q and 2q respectively. Then $u = w_1 \otimes w_1$ and $v = w_2 \otimes w_2$. The cup product $w_1{}^2 \in H^{2q}(S^q \cup_\xi e^{2q})$ is the same as the corresponding cup product in $H^{2q}(\Omega \Sigma S^q)$ and, for q even, it is well known [30] that $w_1{}^2 = 2w_2$.

Hence by the multiplication rule for the tensor product of two rings

$$u^2 = (w_1 \otimes w_1)^2 = w_1^{\,2} \otimes w_1^{\,2}$$
$$= 4w_2 \otimes w_2 = 4\,v.$$

Therefore by Steenrod's definition, the Hopf invariant of γ is 4.

It follows from [25] that when $S^q_\infty \wedge S^q_\infty$ is suspended all the cells are attached trivially and hence $\Sigma\gamma = 0 \in \pi_{4q}(S^{2q+1})$. Therefore by the delicate suspension theorem [34; (3.49)] γ is a multiple of $[\iota_{2q}, \iota_{2q}]$. Since $2q$ is even the Hopf invariant of $[\iota_{2q}, \iota_{2q}]$ is 2 and hence $\gamma = 2\,[\iota_{2q}, \iota_{2q}]$.

Therefore $T = S^{2q} \cup_\gamma e^{4q} \vee S^{3q} \vee S^{3q} \vee S^{4q} \vee S^{4q}$ is the $(5q-1)$-skeleton of $S^q_\infty \wedge S^q_\infty$ and it has the same $(5q-2)$-homotopy type as $\Omega\Sigma S^q \wedge \Omega\Sigma S^q$. This completes the proof of the lemma.

Hence, for q even, there exists a map $k: T \longrightarrow \Omega\Sigma S^q \wedge \Omega\Sigma S^q$ which induces isomorphisms in homotopy in dimensions $\leqslant 5q-2$. Now it is clear that $(e \wedge e)_*$ factors into

$$\pi_n(S^{2q}) \xrightarrow{i_*} \pi_n(S^{2q} \vee_\gamma e^{4q}) \xrightarrow{j_*} \pi_n(T) \xrightarrow{k_*} \pi_n(\Omega\Sigma S^q \wedge \Omega\Sigma S^q)$$

where i and j are the inclusion maps and j_* maps monomorphically into a direct summand.

In the case when q is odd it is clear from the proof of Lemma 2.4 that $\Omega\Sigma S^q \wedge \Omega\Sigma S^q$ has the same $(5q-2)$-homotopy type as a wedge of spheres. Hence the map $(e \wedge e)_*$ is monomorphic in dimensions $\leqslant 5q-2$ and the following proposition follows from Theorems 2.1 and 2.2.

Proposition 2.5. Let $B = S^q \cup_\alpha e^n$, q odd and $n \leqslant 5q-2$. Then wcat $e \leqslant 1$ if and only if wcat $B \leqslant 1$.

In Theorem 5.2 of [14] it is proved that wcat $e \geqslant$ conil B but it is mentioned that an example of strict inequality has not been produced. We will now use an example which occurs later in the above paper to show that the strict inequality can occur.

Example 2.6. Let $B = S^2 \cup_\alpha e^8$ where $\alpha = \eta_2 \circ \nu' \circ \eta_6$ is the generator of order 2 in $\pi_7(S^2)$. Then wcat $e = 2$ and conil $B = 1$.

Proof. The element η_k generates $\pi_{k+1}(S^k)$ and ν' generates $\pi_6(S^3)$. The crude Hopf invariant $\bar{H}(\alpha) = \Sigma\nu' \circ \eta_7 \neq 0 \in \pi_8(S^4)$.

Now $\Sigma \bar{H}(\alpha) = 0 \in \pi_9(S^5)$ and as is mentioned in [14; Theorem 6.1] it follows from Theorem 2.3 that conil B = 1.

By one of the Blakers-Massey theorems [6; Theorem II] the sequence

$$\pi_8(S^7) \xrightarrow{\gamma_*} \pi_8(S^4) \xrightarrow{i_*} \pi_8(S^4 \cup_\gamma e^8)$$

is exact. Since $\pi_8(S^7) = Z_2$ and $\gamma = 2[\iota_4, \iota_4]$ it follows that $\gamma_* = 0$ and hence Ker $i_* = 0$. Now $\mathrm{Ker}(e \wedge e)_* = \mathrm{Ker}(k_* \circ j_* \circ i_*) = 0$ since the kernals of each of the maps i_*, j_* and k_* are zero. Hence $(e \wedge e)_* \bar{H}(\alpha) \neq 0$ and by Theorem 2.2 wcat $e > 1$. But B is a CW-complex with three cells and so by Proposition 1.3 cat B = wcat e = 2.

<u>Example 2.7.</u> Let $B = S^2 \cup_\alpha e^{10}$ where $\alpha = \eta_2 \circ \alpha_1(3) \circ \alpha_1(6)$ is the generator of order 3 in $\pi_9(S^2)$. Then wcat B = 2 and wcat e = 1.

<u>Proof.</u> The element $\alpha_1(k)$ is an element of order 3 in $\pi_{k+3}(S^k)$. Let $H_2 : \pi_{n-1}(S^q) \longrightarrow \pi_{n-1}(S^{2q-1})$ be a Hopf invariant (see Definition 5.1). For q = 2 and $n \geqslant 4$ H_2 is an isomorphism and hence $H_2(\alpha) = \alpha_1(3) \circ \alpha_1(6) \in \pi_9(S^3)$. By Proposition 5.2 (ii) $\bar{H}(\alpha) = \Sigma H_2(\alpha) = \alpha_1(4) \circ \alpha_1(7) \neq 0$ and so

wcat B $=2$ by Theorem 2.1.

Hilton [17; p.195] proves that $[[\iota_4, \iota_4], \iota_4] = \pm \alpha_1(4) \circ \alpha_1(7)$. Therefore $\bar{H}(\alpha) = \pm [[\iota_4, \iota_4], \iota_4] = \mp [\gamma, \iota_4]$ since $\gamma = 2\,[\iota_4, \iota_4]$ and $\bar{H}(\alpha)$ is of order 3. By the naturality of the Whitehead product

$$\begin{aligned} i_*\bar{H}(\alpha) &= \mp\, i_*[\gamma, \iota_4] \\ &= \mp\, [i_*\gamma, i_*\iota_4] \\ &= 0 \in \pi_{10}(S^4 \cup_\gamma e^8) \end{aligned}$$

since $i_*\gamma = 0$. Hence $(e \wedge e)_*\bar{H}(\alpha) = 0$ and wcat $e = 1$ by Theorem 2.2.

3. OPERATIONS ON FIBRE SPACES

In [32] Svarc generalizes the Whitney sum construction and defines the 'sum' of two fibrations over the same base to be a fibration whose fibre is the join of the two original fibres. We give Svarc's definition in terms of the generalized join. We also define a 'product' of two fibrations over different bases to be a fibration over the product of the bases. The fibre is again the join of the original fibres. Theorem 3.4 shows the connection between the 'sum' and the 'product' constructions.

Definition 3.1. [27] For each k, $0 \leq k \leq n$, define the generalized join $J_k(F_1,F_2,..,F_n)$ of the spaces $F_1,F_2,..,F_n$ to be the subset of $CF_1 \times CF_2 \times .. \times CF_n$ consisting of points $(y_1,y_2,..,y_n)$ such that $y_i \in F_i$ for at least k values of i. F_i is embedded as the base of the cone CF_i.

For example $J_1(F_1,F_2,..,F_n)$ is the usual join $F_1 * F_2 * .. * F_n$ and $J_n(F_1,F_2,..,F_n)$ is the product $F_1 \times F_2 \times .. \times F_n$.

Let $\mathcal{B}_i : F_i \longrightarrow E_i \xrightarrow{p_i} B$ $(1 \leq i \leq n)$ be a set of n fibrations over the same base B. Let

$Z_i = \dfrac{E_i \times I \cup B}{(e,o) \sim p_i(e)}$ be the mapping cylinder of p_i

and let $\overline{p_i}: Z_i \longrightarrow B$ be the projection.

<u>Definition 3.2.</u> For $o \leqslant k \leqslant n$, the <u>generalized sum</u> of the fibrations $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n$ is the fibration

$$S_k(\mathcal{B}_1, \ldots, \mathcal{B}_n): J_k(F_1, \ldots, F_n) \longrightarrow E \xrightarrow{p} B$$

where the total space

$$E = \left\{ (z_1, \ldots, z_n) \in Z_1 \times \ldots \times Z_n \,\middle|\, \overline{p_1}(z_1) = \ldots = p_n(z_n) \text{ and } z_i \in E_i \text{ for at least } k \text{ values of } i \,. \right\}$$

The projection $p: E \longrightarrow B$ is defined by

$p(z_1, \ldots, z_n) = \overline{p_1}(z_1)$.

If $k = 1$ we write $S_1(\mathcal{B}_1, \ldots, \mathcal{B}_n) = \mathcal{B}_1 + \ldots + \mathcal{B}_n$ and this is then the sum as defined by Svarc and also by I. M. Hall [16]. It is clear that this sum is commutative and associative [16].

Now let $\mathcal{B}_i: F_i \longrightarrow E_i \xrightarrow{p_i} B_i$ $(1 \leqslant i \leqslant n)$ be a set of n fibrations with any bases and again let Z_i be the mapping cylinder of p_i.

<u>Definition 3.3.</u> For $o \leqslant k \leqslant n$, the <u>generalized product</u> of the fibrations $\mathcal{B}_1, \ldots, \mathcal{B}_n$ is the fibration

$$T_k(\mathcal{B}_1,\dots,\mathcal{B}_n):\ J_k(F_1,\dots,F_n)\longrightarrow E \xrightarrow{p} B_1\times\dots\times B_n$$

where the total space

$$E = \{(z_1,\dots,z_n)\in Z_1\times\dots\times Z_n \mid z_i\in E_i \text{ for at least } k \text{ values of } i\ .\}$$

The projection is defined by

$$p(z_1,\dots,z_n) = (p_1 z_1,\dots,p_n z_n).$$

If $k = 1$ we write $T_1(\mathcal{B}_1,\dots,\mathcal{B}_n) = \mathcal{B}_1\times\dots\times\mathcal{B}_n$. (Note that Svarc's definition of the product is different from this; in fact Svarc defines the product of fibrations with the same base to be $S_n(\mathcal{B}_1,\dots,\mathcal{B}_n)$ which has fibre $F_1\times\dots\times F_n$.)

Again let $\mathcal{B}_i$ $(1\leqslant i\leqslant n)$ be a set of n fibrations with any bases and suppose we have n maps $g_i:B\longrightarrow B_i$. Let $g_i{}^*\mathcal{B}_i$ be the fibration induced from $\mathcal{B}_i$ by g_i. Let $\Delta':B\longrightarrow B_1\times\dots\times B_n$ be defined by $\Delta'(b) = (g_1(b),\dots,g_n(b))$.

<u>Theorem 3.4.</u>

$$S_k(g_1{}^*\mathcal{B}_1,\dots,g_n{}^*\mathcal{B}_n) = \Delta'^*T_k(\mathcal{B}_1,\dots,\mathcal{B}_n).$$

<u>Proof.</u> The fibration $g_i{}^*\mathcal{B}_i:F_i\longrightarrow E_i{}^*\xrightarrow{p_i{}^*} B$ has total space $E_i{}^* = \{(e_i,b)\in E_i\times B \mid p_i(e_i) = g_i(b)\}$ and the mapping cylinder of $p_i{}^*$ is

$$Z_i^* = \left\{ (e_i,b,t) \in E_i \times B \times I \;\middle|\; p_i(e_i) = g_i(b) \text{ and } (e_i,b,0) = (e_i',b,0) \right\}.$$

The sum $S_k(g_1{}^*\mathcal{B}_1,\ldots,g_n{}^*\mathcal{B}_n): J_k(F_1,\ldots,F_n) \longrightarrow S \xrightarrow{\hat{p}} B$ has total space

$$S = \left\{ (e_1,b,t_1,\ldots,e_n,b,t_n) \in Z_1^* \times \ldots \times Z_n^* \;\middle|\; t_i = 1 \text{ for at least } k \text{ values of } i \right\}.$$

(3.5) Hence $S = \left\{ (e_1,t_1,\ldots,e_n,t_n,b) \in E_1 \times I \times \ldots \times E_n \times I \times B \;\middle|\; t_i = 1 \text{ for at least } k \text{ values of } i,\ p_i(e_i) = g_i(b) \text{ and } (\ldots,e_i,0,\ldots) = (\ldots,e_i',0,\ldots) \text{ if } p_i(e_i) = p_i(e_i') \right\}.$

Also $\hat{p}(e_1,t_1,\ldots,e_n,t_n,b) = b.$

Let $T_k(\mathcal{B}_1,\ldots,\mathcal{B}_n)$ be the fibration

$$J_k(F_1,\ldots,F_n) \longrightarrow E \xrightarrow{p} B_1 \times \ldots \times B_n$$

and $\triangle'^* T_k(\mathcal{B}_1,\ldots,\mathcal{B}_n)$ be

$$J_k(F_1,\ldots,F_n) \longrightarrow T \xrightarrow{\bar{p}} B.$$

Then $E = \left\{ (e_1,t_1,\ldots,e_n,t_n) \in Z_1 \times \ldots \times Z_n \;\middle|\; t_i = 1 \text{ for at least } k \text{ values of } i \right\}.$

(3.6) Therefore $T = \left\{ (e_1,t_1,\ldots,e_n,t_n,b) \in Z_1 \times \ldots \times Z_n \times B \;\middle|\; t_i = 1 \text{ for at least } k \text{ values of } i \text{ and } p_i(e_i) = g_i(b) \right\}$

and $\bar{p}(e_1,t_1,\ldots,e_n,t_n,b) = b.$

It is clear from the definition of the mapping cylinder of p_i and (3.5) and (3.6) that $S = T$ and $\hat{p} = \bar{p}$. This proves the theorem.

We now give some examples of these constructions.

(3.7) The Whitney sum. If $\mathcal{B}_i$ $(i = 1,2)$ is the sphere bundle associated to a vector bundle $\mathcal{V}_i$, then $\mathcal{B}_1 + \mathcal{B}_2$ is the sphere bundle associated to the Whitney sum $\mathcal{V}_1 \oplus \mathcal{V}_2$ of the vector bundles.

(3.8) The tangent sphere bundle of a product manifold. Let $\mathcal{B}_M : D^m \longrightarrow D_M \longrightarrow M$ be the unit disc bundle over a manifold M^m and let $\partial\mathcal{B}_M : S^{m-1} \longrightarrow \partial D_M \xrightarrow{p_M} M$ be the tangent sphere bundle. Similarly let $\mathcal{B}_N$ and $\partial\mathcal{B}_N$ be the unit disk bundle and tangent sphere bundle over a manifold N^n.

Then the fibration $\partial\mathcal{B}_M \times \partial\mathcal{B}_N$ is equivalent to the tangent sphere bundle $\partial\mathcal{B}_{M\times N}$.

To see this, let the mapping cylinder of p_M be $Z_M = \left\{(m,\hat{v},t) \in M \times V^m \times I \;\middle|\; \|v\| = 1 \text{ and } (m,\hat{v},o) = (m,\hat{v}',o)\right\}$ where V^m is the m dimensional vector space. Define a map $\lambda : Z_M \longrightarrow D_M$ by $\lambda(m,\hat{v},t) = (m,t\hat{v})$. This is clearly a homeomorphism with inverse

$$\lambda^{-1}(m,\hat{v}) = \begin{cases} (m, \frac{\hat{v}}{\|\hat{v}\|}, \|\hat{v}\|) & \text{if } \|\hat{v}\| \neq 0 \\ (m,\hat{v}',o) & \text{if } \|\hat{v}\| = 0 . \end{cases}$$

Hence the total space of $\partial\mathcal{B}_M \times \partial\mathcal{B}_N$ is

$\partial D_M \times D_N \cup D_M \times \partial D_N = \partial (D_M \times D_N)$ and it will follow that the fibration $\partial \mathcal{B}_M \times \partial \mathcal{B}_N$ is equivalent to $\partial \mathcal{B}_{M \times N}$.

(3.9) The fibration $\Omega B * \Omega B \longrightarrow \Sigma \Omega B \longrightarrow B$.

Let $\mathcal{B}: \Omega B \longrightarrow PB \longrightarrow B$ be the standard fibration in which PB is the space of paths in B starting at $*$. Svarc [32; Theorem 21] proves that the fibration $\mathcal{B} + \mathcal{B}$ is homotopically equivalent to the fibration $\Omega B * \Omega B \longrightarrow \Sigma \Omega B \xrightarrow{e'} B$ studied by Barcus and Meyer [1].

The product $\mathcal{B} \times \mathcal{B}$ is homotopically equivalent to the fibration obtained from the inclusion map $j: B \vee B \longrightarrow B \times B$; i.e. the fibration

$$E(B \times B; B \vee B, *) \longrightarrow E(B \times B; B \vee B, B \times B) \xrightarrow{q} B \times B.$$

To show this let the total space of $\mathcal{B} \times \mathcal{B}$ be $E = \{(\alpha, \tau_1, \beta, \tau_2) \in Z \times Z \mid \tau_1 \text{ or } \tau_2 = 1\}$ where Z is the mapping cylinder of $PB \longrightarrow B$. Define a map $\mu: E \longrightarrow E(B \times B; B \vee B, B \times B)$ by $\mu(\alpha, \tau_1, \beta, \tau_2) = (\bar{\alpha}, \bar{\beta})$ where $\bar{\alpha}(t) = \alpha(1-\tau_1+t\tau_1)$ and $\bar{\beta}(t) = \beta(1-\tau_2+t\,\tau_2)$. Since $q \circ \mu: E \longrightarrow B \times B$ is the same as the fibre map of $\mathcal{B} \times \mathcal{B}$, μ induces a map between the fibres $\mu': \Omega B * \Omega B \longrightarrow E(B \times B; B \vee B, *)$. By the arguments on p. 135 of [14], if B is simply connected, then μ is a homotopy equivalence. Hence by the homotopy exact sequence for a fibration, the 5-lemma and Whitehead's Theorem [36; Theorem 1] $\mathcal{B} \times \mathcal{B}$ is homotopic to the fibration obtained from $j: B \vee B \longrightarrow B \times B$.

In this case it follows from Theorem 3.4 that $\mathcal{B} + \mathcal{B} = \Delta^*(\mathcal{B} \times \mathcal{B})$. This is equivalent to the fact that in the following commutative diagram

$$\begin{array}{ccccc} \Omega B * \Omega B & \longrightarrow & \Sigma \Omega B & \xrightarrow{\quad e' \quad} & B \\ \| & & \big\downarrow {\scriptstyle (e \vee e) \circ \lambda} & & \big\downarrow {\scriptstyle \Delta} \\ \Omega B * \Omega B & \longrightarrow & B \vee B & \xrightarrow{\quad j \quad} & B \times B \end{array}$$

the fibration $\Omega B * \Omega B \longrightarrow \Sigma \Omega B \longrightarrow B$ is induced by Δ from the fibration obtained from the map $j: B \vee B \longrightarrow B \times B$.

This can be generalized to prove that $\mathcal{B} \times .. \times \mathcal{B}$ (k+1 times) is equivalent to the fibration obtained from the map $j: T_1^{k+1} \longrightarrow B^{k+1}$.

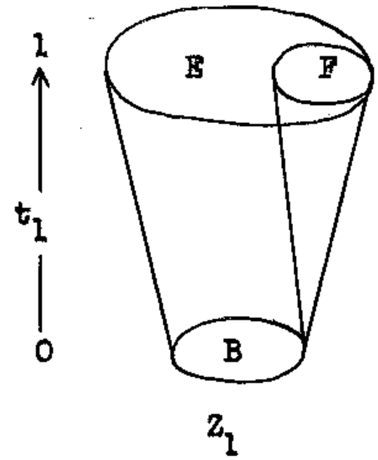
(3.10) Extending fibrations. In [12; §1] Ganea describes a process for extending a fibration $\mathfrak{F}: F \xrightarrow{i} E \xrightarrow{p} B$ to another fibration $\mathfrak{F}'$ as follows.

$$\begin{array}{ccccc}
\mathfrak{F}: F & \xrightarrow{\quad i \quad} & E & \xrightarrow{\quad p \quad} & B \\
 & & \downarrow & & \| \\
\mathfrak{F}': F * \Omega B & \xrightarrow{\qquad} & E \cup CF & \xrightarrow{\quad r \quad} & B
\end{array}$$

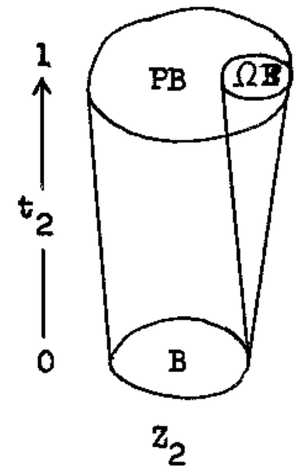
Let $E \cup CF$ be the cofibre of i. Then we can extend p to a map $r: E \cup CF \longrightarrow B$ by mapping CF to the base point. Ganea [12; Theorem 1.1] proves that when r is converted into a fibration $\mathfrak{F}'$ its fibre is weakly homotopy equivalent to $F * \Omega B$.

We will show that $\mathfrak{F}'$ is equivalent to the sum $\mathfrak{F} + \mathfrak{B}$.

Let E' and $\overline{E}$ be the total spaces of $\mathfrak{F}'$ and $\mathfrak{F} + \mathfrak{B}$ with projections $\overline{p}: \overline{E} \longrightarrow B$ and $p': E' \longrightarrow B$. Let Z_1 and Z_2 be the mapping cylinders of $p: E \longrightarrow B$ and $PB \longrightarrow B$.



Z_1



Z_2

Then $\overline{E} \subset E \times Z_2 \cup Z_1 \times PB$ and

$E' = \{(te, \beta) \in E \cup CF \times B^I \mid \beta(1) = p(e)\}$.

Let $\lambda : \{(e, \beta) \in E \times B^I \mid \beta(0) = p(e)\} \longrightarrow E^I$ be a path lifting map for $\mathcal{F}$ [10]. Define a map $w : \overline{E} \longrightarrow E'$ by $w(t_1 e, t_2 \beta) = (t_1 \lambda(e, -\beta_{t_2})(1), \beta_{t_2})$ where $\beta_{t_2}(\tau) = \beta(t_2 \tau)$ and β is a path in PB which ends at *.

Then $\overline{p}(t_1 e, t_2 \beta) = p(e) = \beta(0)$ and

$$p' w(t_1 e, t_2 \beta) = \beta_{t_2}(0) = \beta(0).$$

Hence $p' w = \overline{p} : \overline{E} \longrightarrow B$.

Now the map w induces a map between the fibres of $\mathcal{F} + \mathcal{B}$ and $\mathcal{F}'$ which is precisely the map $w_3 \circ w_2$

in the proof of Theorem 1.1 of [12]. It is proved there that the map is a homotopy equivalence and hence by the homotopy sequence for a fibration, the 5-lemma and Whitehead's Theorem [36; Theorem 1] the fibrations $\mathcal{F}'$ and $\mathcal{F} + \mathcal{B}$ are equivalent.

4. GANEA'S DEFINITION OF CATEGORY

Again let $\mathcal{B}: \Omega B \longrightarrow PB \longrightarrow B$ be the standard fibration and, for $k \geqslant 0$, let $\mathcal{F}_k : F_k \longrightarrow E_k \xrightarrow{p_k} B$ be the fibration defined by $\mathcal{F}_k = \mathcal{B} + \mathcal{B} + .. + \mathcal{B}$ (k+1 copies). Then $F_k = \Omega B * \Omega B * .. * \Omega B$ (k+1 copies). Alternatively $\mathcal{F}_k$ could be defined inductively by letting $\mathcal{F}_0 = \mathcal{B}$ and then obtaining $\mathcal{F}_k$ by extending $\mathcal{F}_{k-1}$ (See (3.10)).

The following definition of category is due to Ganea (See Definition 6.1 of [12] for the dual definition).

__Definition 4.1.__ __G-cat B__ is the least integer $k \geqslant 0$ for which there exists a map $r: B \longrightarrow E_k$ such that $p_k \circ r \simeq 1$; if no such integer exists G-cat $B = \infty$.

When $p_k : E_k \longrightarrow B$ is converted into a cofibre map let C_k be its cofibre and $q_k : B \longrightarrow C_k$ be its induced map.

__Definition 4.2.__ __G-wcat B__ is the least integer $k \geqslant 0$ for which $q_k \simeq 0$; if no such integer exists G-wcat $B = \infty$.

It is clear that G-cat $B \geqslant$ G-wcat B.

The following theorem is also proved in

[13; Proposition 2.2] directly from the classical Lusternik-Schnirelmann definition of category, instead of from G. W. Whitehead's definition used here.

Theorem 4.3. G-cat B = cat B.

Proof. By Theorem 3.4 $\mathcal{F}_k = \mathcal{B} + \dots + \mathcal{B} = \Delta^*(\mathcal{B} \times \dots \times \mathcal{B})$ and hence by (3.9) is induced by $\Delta : B \longrightarrow B^{k+1}$ from the fibration obtained from $j : T_1^{\,k+1} \longrightarrow B^{k+1}$. Hence there exists a map $u : E_k \longrightarrow T_1^{\,k+1}$ such that the middle square in the following diagram is homotopy commutative.

$$\begin{array}{ccccccc}
F_k & \longrightarrow & E_k & \xrightarrow{p_k} & B & \xrightarrow{q_k} & C_k \\
\| & & \big\downarrow u & & \big\downarrow \Delta & & \big\downarrow \Delta' \\
F_k & \longrightarrow & T_1^{\,k+1} & \xrightarrow{j} & B^{k+1} & \xrightarrow{q} & B^{(k+1)}
\end{array}$$

Suppose G-cat $B \leqslant k$ and so $\mathcal{F}_k$ has a cross-section $r : B \longrightarrow E_k$. Let $\varphi = u \circ r : B \longrightarrow T_1^{\,k+1}$ and then $j \circ \varphi \simeq j \circ u \circ r \simeq \Delta \circ p_k \circ r \simeq \Delta$. Hence cat $B \leqslant k$.

Conversely suppose cat $B \leqslant k$ and there exists a map $\varphi : B \longrightarrow T_1^{\,k+1}$ and a homotopy $\Psi_t : B \longrightarrow B^{k+1}$

such that $\Psi_0 = \Delta$ and $\Psi_1 = j \circ \varphi$. Define the map $r : B \longrightarrow E(B^{k+1}; \Delta B, T_1^{k+1})$ by $r(b)(t) = \Psi_t(b)$. This is a cross-section to the fibration induced by Δ from j. Hence it induces a cross-section of $\mathcal{F}_k$ and so G-wcat $B \leqslant k$.

<u>Theorem 4.4.</u> G-wcat $B \geqslant$ wcat B.

<u>Proof.</u> Since $j \circ u \simeq \Delta \circ p_k$ there is an induced map $\Delta' : C_k \longrightarrow B^{(k+1)}$ between the cofibres of p_k and j [29; (2.2)] such that $q \circ \Delta \simeq \Delta' \circ q_k$.

Now suppose G-wcat $B = k$, then $q_k \simeq 0$ and so $q \circ \Delta \simeq 0$. Hence wcat $B \leqslant k$.

We collect together, from the above two theorems and (1.8), the results on the relationships between the various invariants in the following theorem.

<u>Theorem 4.5.</u> Let B have the homotopy type of a simply connected CW-complex, then

cat B=G-cat $B \geqslant$ G-wcat $B \geqslant$ wcat $B \geqslant$ wcat $e \geqslant$ conil $B \geqslant \cup$-long B

and furthermore all the inequalities can occur.

Examples 2.7, 2.6, and 1.12 show that the last three inequalities can occur and Examples 5.6 and 5.7 in the next chapter will show that the first two can occur.

5. WEAK CATEGORY AND THE COMPOSITE HOPF INVARIANT

In this chapter we recall the properties of the various Hopf invariants that we will need and give a criterion for G-wcat $B \leq 1$ in terms of a composite Hopf invariant. We will then find examples of spaces which distinguish G-wcat B from wcat B and cat B.

Throughout this chapter we will take B to be the cofibre of a map $\alpha : S^{n-1} \longrightarrow Y$. We are interested in the fibration $\mathcal{F}_1$, which by (3.9) is equivalent to the fibration $\Omega B * \Omega B \longrightarrow \Sigma\Omega B \xrightarrow{e'} B$ where e' is the evaluation map.

We will now define a composite higher Hopf invariant in order to use it to approximate $\Sigma\Omega B$ by a simpler space.

Let D be the infinite wedge $\bigvee_{i \geq 1} S^{i(q-1)+1}$ and let $\tau_k : D \longrightarrow S^{k(q-1)+1}$ be the projection onto the k-th factor. Let $D' = \bigvee_{i \geq 2} S^{i(q-1)+1}$ and $p' : D \longrightarrow D'$ be the map which shrinks S^q to the base point.

Then D is homotopic to $\Sigma\Omega S^q$ and ΣS^{q-1}_∞ [25] and we can fix a homotopy equivalence $\psi : (\Sigma S^{q-1}_\infty, \Sigma S^{q-1}) \longrightarrow (D, S^q)$ which is the identity on S^q. This can be done by using the James' maps

$S_{\infty}^{q-1} \longrightarrow S_{\infty}^{k(q-1)}$ [19; p. 24]. Let $\theta : S_{\infty}^{q-1} \longrightarrow \Omega S^q$ be the canonical weak homotopy equivalence of the reduced product complex [21]. Denote the suspension homomorphism by Σ and the Hurewicz isomorphism by $\rho : \pi_{n-1}(S^q) \longrightarrow \pi_{n-2}(\Omega S^q)$.

Definition 5.1. The composite higher Hopf invariants $H : \pi_{n-1}(S^q) \longrightarrow \pi_{n-1}(D)$ and $H' : \pi_{n-1}(S^q) \longrightarrow \pi_{n-1}(D')$ are defined by $H = \psi_* \circ \Sigma \circ \theta_*^{-1} \circ \rho$ and $H' = p'_* \circ H$.

For $k \geqslant 1$, the higher Hopf invariants $H_k : \pi_{n-1}(S^q) \longrightarrow \pi_{n-1}(S^{k(q-1)+1})$ are defined by $H_k = \tau_{k*} \circ H$.

In the next proposition we recall from (3.10) and Theorem (3.19) of [19] the properties of these Hopf invariants we will require. We also state the connections between these Hopf invariants and the crude Hopf invariant $\overline{H} : \pi_{n-1}(S^q) \longrightarrow \pi_n(S^q \wedge S^q)$ and the delicate Hopf invariant $\mathcal{H} : \pi_{n-1}(S^q) \longrightarrow \pi_n(S^q \times S^q, S^q \vee S^q)$. Part (iii) follows from (3.14) of [19] and part (iv) from the work of Barratt [2] who has found the relationships between the higher Hopf invariants of James and those of Hilton.

Proposition 5.2.

(i) $H_1 = 1$, the identity homomorphism.

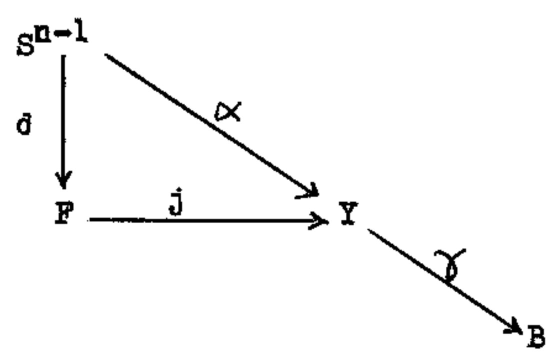
(ii) $H(\xi \circ \Sigma \eta) = H(\xi) \circ \Sigma \eta$, where $\xi \in \pi_m(S^q)$ and $\eta \in \pi_{n-2}(S^{m-1})$.

(iii) $\overline{H} = \Sigma H_2$

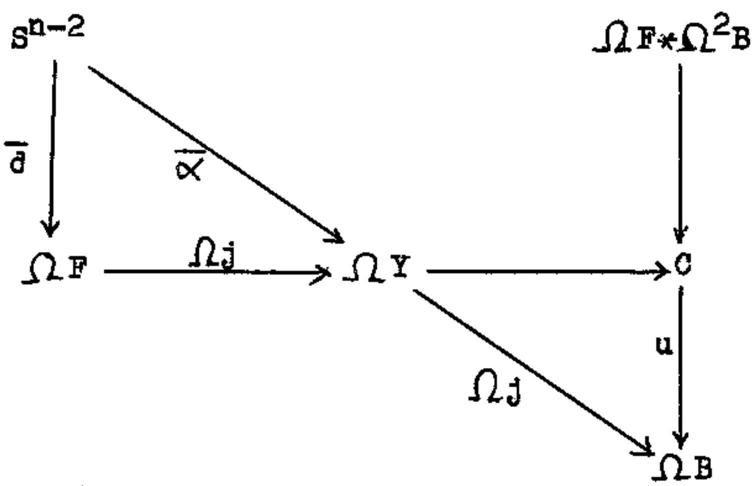
(iv) If $\mathcal{H}(\alpha) = 0$ then $H_k(\alpha) = 0$ for $k \geq 2$.

Proposition 5.3. Let $S^{n-1} \xrightarrow{\alpha} Y \xrightarrow{\gamma} B$ be a cofibration in which Y is $(q-1)$-connected, $(n-1) \geq q \geq 3$. Then there exists an $(n+q-2)$-connected map $m: \Sigma \Omega Y \cup_\beta e^n \longrightarrow \Sigma \Omega B$ where $\beta = \Sigma \overline{\alpha}$ and $\overline{\alpha}: S^{n-2} \longrightarrow \Omega Y$ is the adjoint of α.

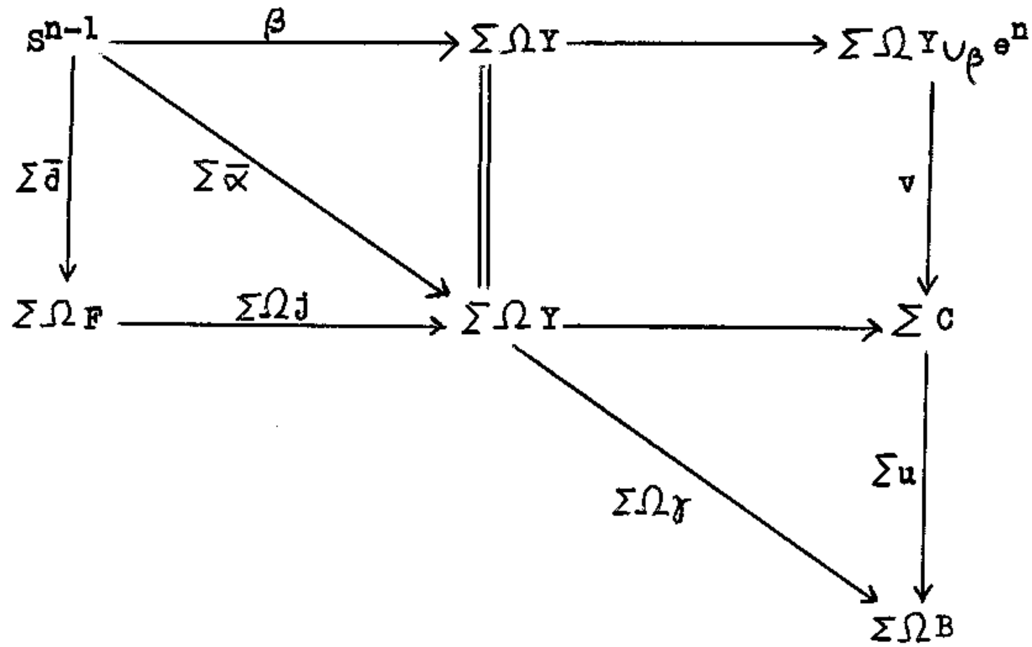
Proof. Convert γ into a fibre map, let F be the fibre and $j: F \longrightarrow Y$ be induced from the inclusion map of the fibre. Lift α to a map $d: S^{n-1} \longrightarrow F$ such that $\alpha \simeq j \circ d$; then by Lemma 3.1 of [12] d is $(n+q-3)$-connected



Let C be the cofibre of $\Omega j : \Omega F \longrightarrow \Omega Y$ and extend $\Omega\lambda$ to a map $u : C \longrightarrow \Omega B$. By Theorem 1.1 of [12] the fibre of u is homotopic to $\Omega F * \Omega^2 B$; hence u is (n+q-3)-connected.



Let $\beta = \Sigma\bar{\alpha}$, then $\beta \simeq \Sigma\Omega j \circ \Sigma\bar{d}$ and in the following diagram the horizontal sequences are cofibrations and v is induces from the maps between these cofibrations [29; (2.2)].



Since $\Sigma\bar{\partial}$ is $(n+q-3)$-connected, by applying the 5-lemma to the homology exact sequence of the above cofibrations we see that v is $(n+q-2)$-connected.

Let $m = \Sigma u \circ v : \Sigma\Omega Y \cup_\beta e^n \longrightarrow \Sigma\Omega B$ and then the proposition follows.

<u>Remark 5.4.</u> If $Y = S^q$ in the above proposition so that $B = S^q \cup_\alpha e^n$ then the map $\beta = \Sigma\bar{\alpha} \in \pi_{n-1}(\Sigma\Omega S^q)$. But $H(\alpha) = \psi_* \circ \Sigma \circ \theta_*^{-1}\bar{\alpha} = \psi_* \circ (\Sigma\theta)_*^{-1}\Sigma\bar{\alpha} = \psi_* \circ (\Sigma\theta)_*^{-1}\beta$ and $\psi_* \circ (\Sigma\theta)_*^{-1} : \pi_{n-1}(\Sigma\Omega S^q) \longrightarrow \pi_{n-1}(D)$ is an isomorphism. Hence in the above proposition we can consider β to be $H(\alpha)$ and m to have domain $D \cup_\beta e^n$.

<u>Theorem 5.5.</u> If $B = S^q \cup_\alpha e^n$, $n-1 \geq q \geq 3$, then G-wcat $B \leq 1$ if and only if $\Sigma H'(\alpha) = 0 \in \pi_n(\Sigma D')$.

<u>Proof.</u> Let $m : D \cup_\beta e^n \longrightarrow \Sigma\Omega B$ be the map refered to in the previous remark and let C_1' be the cofibre of the map $e' \circ m$.

$$\begin{array}{ccccc}
D \cup_\beta e^n & \xrightarrow{\ e' \circ m\ } & B & \xrightarrow{\ q_1'\ } & C_1' \\
\Big\downarrow m & & \Big\| & & \Big\downarrow m_1 \\
\Sigma\Omega B & \xrightarrow{\ e'\ } & B & \xrightarrow{\ q_1\ } & C_1
\end{array}$$

In the above diagram m induces a map of cofibres $m_1 : C_1' \longrightarrow C_1$. By applying the 5-lemma to the homology exact sequence of the above cofibrations we see that m_1 is $(n+q-1)$-connected.

Now $C_1' = B \cup C(D \cup_\beta e^n)$
$= (S^q \cup_\alpha e^n) \cup C((S^q \vee D') \cup_\beta e^n)$

and by homology considerations $e' \circ m$ maps S^q onto S^q with degree 1. Therefore since the embedding $S^q \subset C_1'$ can be pulled back to the cofibre $(S^q \vee D') \cup_\beta e^n$ it is nullhomotopic. Hence shrinking S^q to a point

$C_1' \simeq C_\epsilon = S^n \cup_\epsilon C(D' \cup_\delta e^n)$ where $\delta = p' \circ \beta = H'(\alpha)$ and C_ϵ is the cofibre of the map $\epsilon : D \cup_\delta e^n \longrightarrow S^n$ induced from $e' \circ m$. Since $e' \circ m$ maps D into S^q when we shrink S^q to a point ϵ maps D' to the base point.

$$\begin{array}{ccccc} D \cup_\beta e^n & \xrightarrow{e' \circ m} & S^q \cup_\alpha e^n & \xrightarrow{q_1'} & C_1' \\ \downarrow & & \downarrow & & \downarrow \simeq \\ D' \cup_\delta e^n & \xrightarrow{\epsilon} & S^n & \xrightarrow{\bar{q}} & C_\epsilon \end{array}$$

We will now show that C_ϵ and hence C_1' is homotopic to $\Sigma D'$. In dimension n, $m_* : H_n(D \cup_\beta e^n) \longrightarrow H_n(\Sigma \Omega B)$ is an isomorphism so that ϵ maps the n-cell onto S^n with degree ± 1. Hence if the degree of ϵ on the n-cell is $+1$, ϵ is homotopic to the map ϵ' which occurs in the following cofibration sequence for δ:

$$S^{n-1} \xrightarrow{\delta} D' \longrightarrow D' \cup_\delta e^n \xrightarrow{\epsilon'} S^n \xrightarrow{\Sigma \delta} \Sigma D' \longrightarrow \cdots \quad .$$

By [29; Satz 5] C_ϵ is homotopic to $\Sigma D'$ and the inclusion map $\bar{q} : S^n \longrightarrow C_\epsilon$ is homotopic to $\Sigma \delta$. If the degree on the n-cell is -1 then $\bar{q} : S^n \longrightarrow C_\epsilon$ is homotopic to $-\Sigma \delta$.

Let $\widetilde{\alpha}$ be the characteristic map of the n-cell in B. Factor q_1 through C_1' by means of q_1'.

$$\begin{array}{ccccc} (CS^{n-1}, S^{n-1}) & \longrightarrow & (B, S^q) & \longrightarrow & (C_1, *) \\ & & & {\scriptstyle q_1'} \searrow \quad \nearrow {\scriptstyle m_1} & \\ & & & (C_1', *) & \end{array}$$

Let $\varphi \in \pi_n(C_1)$ and $\varphi' \in \pi_n(C_1')$ be the elements represented by $q_1 \circ \widetilde{\alpha}$ and $q_1' \circ \widetilde{\alpha}$ respectively. Since the inclusion map $\bar{q}: S^n \longrightarrow C_\epsilon$ is obtained from the map $q_1': S^q \cup e^n \longrightarrow C_1'$ by shrinking S^q to a point, $\bar{q}$ is in the homotopy class φ'. Hence

$\varphi' = \bar{q} = \pm \Sigma \delta = \pm \Sigma H'(\alpha) \in \pi_n(\Sigma D')$.

Now $\varphi = m_{1*}\varphi'$ where m_{1*} is an isomorphism in dimension n. If $n-1 \geqslant q \geqslant 3$, C_1 has no cells in positive dimensions less than $q+1$ and it follows that $q_1 \simeq 0$ if and only if $\varphi = 0$.

Hence the following five statements are equivalent:

(i) G-wcat $B \leqslant 1$

(ii) $q_1 \simeq 0$

(iii) $\varphi = 0 \in \pi_n(C_1)$

(iv) $\varphi' = 0 \in \pi_n(C_1') \approx \pi_n(\Sigma D')$

(v) $\Sigma H'(\alpha) = 0 \in \pi_n(\Sigma D')$.

Example 5.6. Let $B = S^3 \cup_\alpha e^{18}$ where $\alpha = \varepsilon_3 \circ \nu_{11} \circ \nu_{14} \in \pi_{17}(S^3)$ is an element of order 2, then cat $B = 2$ and G-wcat $B = 1$.

Proof. Recall from Chapter 6 of [33] that the element ε_3 of order 2 is the generator of $\pi_{11}(S^3)$ and is defined by the secondary composition $\{\eta_3, \Sigma\nu', \nu_7\}_1$. The element $\nu_n \in \pi_{n+3}(S^n)$ is the generator of order 8 in the stable 3-stem. Since ν_{11} and ν_{14} are both suspensions it follows from Proposition 5.2 that $H'(\alpha) = H'(\varepsilon_3) \circ \nu_{11} \circ \nu_{14}$.

Now $H'(\varepsilon_3) \in \pi_{11}(S^5 \vee S^7 \vee S^9 \vee S^{11})$ which by Theorem A of [18] is isomorphic to the direct sum decomposition

$$\pi_{11}(S^5) \oplus \pi_{11}(S^7) \oplus \pi_{11}(S^9) \oplus \pi_{11}(S^{11}) \oplus \pi_{11}(S^{11}).$$

By the definition of H_k the projections of $H'(\varepsilon_3)$ on the first and third summands are $H_2(\varepsilon_3) \in \pi_{11}(S^5)$ and $H_4(\varepsilon_3) \in \pi_{11}(S^9)$. The projections on the other summands are zero since $\pi_{11}(S^7) = 0$ and $\pi_{11}(S^{11}) = \mathbb{Z}$.

Now by (6.1) of [33] $H_2(\varepsilon_3) = \nu_5 \circ \nu_8$ and by a proof similar to that of [33; (2.3)] we see that

$$H_4(\varepsilon_3) \subset \{H_4(\eta_3), \Sigma\nu', \nu_7\}_1 = 0$$

since the coset consists of a single element. Thus the only non-zero component of $H'(\alpha)$ is

$H_2(\alpha) = \nu_5 \circ \nu_8 \circ \nu_{11} \circ \nu_{14} \in \pi_{17}(S^5)$. From the information on the 12-stem obtained in the proof of (7.6) of [33] we see that $\Sigma H_2(\alpha) = 0$ and hence by Proposition 5.2 (iv) we conclude that $\mathcal{H}(\alpha) \neq 0$ while $\Sigma H'(\alpha) = 0$.

Therefore cat B = 2 by Theorem 2.1 while G-wcat B = 1 by Theorem 5.5.

Example 5.7. Let $B = S^3 \cup_\alpha e^{15}$ where $\alpha = \alpha_3(3)$ is an element of order 3 in $\pi_{14}(S^3)$, then G-wcat B = 2 and wcat B = 1.

Proof. The crude Hopf invariant $\overline{H}(\alpha)$ lies in $\pi_{15}(S^6)$ which contains no elements of order 3. Hence $\overline{H}(\alpha) = 0$ and wcat B = 1 by Theorem 2.1.

But by (13.10) of [33] $H_3(\alpha) = x.\alpha_2(7) \in \pi_{14}(S^7)$ for some $x \not\equiv 0 \pmod 3$. Therefore $\Sigma H_3(\alpha) = x.\alpha_2(8)$ which is non-zero. Hence $\Sigma H'(\alpha) \neq 0$ and by Theorem 5.5 G-wcat B = 2.

6. COCATEGORY

In this chapter we define the invariants cocategory and weak cocategory which are duals, in the sense of Eckmann-Hilton, to G-cat and G-wcat.

We find the relationships between these invariants and the other dual invariants, nilpotency and Whitehead product length. We give examples of spaces which distinguish all the invariants except cocat from wcocat. In a later chapter Example 8.5 will distinguish these latter two.

For any space A construct the ladder of cofibrations

$$\mathcal{C}_k : A \xrightarrow{e_k} M_k \xrightarrow{f_k} N_k$$

as follows. Let $\mathcal{C}_0$ be the standard cofibration in which $M_0 = CA$, the reduced cone over A and $N_0 = \sum A$, the reduced suspension of A. Suppose inductively that $\mathcal{C}_k$ has been defined. Let M'_{k+1} be the fibre of f_k and let $e'_{k+1} : A \longrightarrow M'_{k+1}$ lift e_k. Convert e'_{k+1} into a cofibre map $e_{k+1} : A \longrightarrow M_{k+1}$ where M_{k+1} is the reduced mapping cylinder of e'_{k+1}. Let N_{k+1} be the cofibre of e_{k+1} and $f_{k+1} : M_{k+1} \longrightarrow N_{k+1}$ the identification map.

When e_k is converted into a fibration let the fibre be D_k with projection $d_k : D_k \longrightarrow A$.

$$\begin{array}{ccccccc}
\mathcal{C}_0: N_0 & \xleftarrow{\;f_0\;} & M_0 & \xleftarrow{\;e_0\;} & A & \xleftarrow{\;\partial_0\;} & D_0 \\
 & & \uparrow & & \| & & \\
\mathcal{C}_1: N_1 & \xleftarrow{\;f_1\;} & M_1 & \xleftarrow{\;e_1\;} & A & \xleftarrow{\;\partial_1\;} & D_1 \\
 & & \uparrow & & \| & & \\
 & & \vdots & & \vdots & &
\end{array}$$

Definition 6.1. The cocategory of A, cocat A, is the least integer $k \geqslant 0$ for which there exists a map $r: M_k \longrightarrow A$ such that $r \circ e_k \simeq 1$; if no such integer exists cocat $A = \infty$.

Definition 6.2. The weak cocategory of A, wcocat A, is the least integer $k \geqslant 0$ for which $\partial_k \simeq 0$; if no such integer exists wcocat $A = \infty$.

Since $e_1: A \longrightarrow M_1$ is homotopic to the natural embedding $e: A \longrightarrow \Omega\Sigma A$ it follows from Theorem 1.8 of [21] that cocat $A \leqslant 1$ if and only if A is an H-space.

Let $\varphi_{k+1}: (\Omega A)^{(k+1)} \longrightarrow \Omega A$ be the commutator map with respect to the loop space multiplication on ΩA, as defined in [12; §6].

Definition 6.3. The nilpotency class of the loop space of A, nil A, is the least integer $k \geq 0$ for which $\varphi_{k+1} \simeq 0$; if no such integer exists nil $A = \infty$.

Definition 6.4. The Whitehead product length of A, W-long A, is the least integer $k \geq 0$ such that $[[\cdot\cdot[[\alpha_1, \alpha_2], \alpha_3]\cdot\cdot], \alpha_{k+1}] = 0$ for all $\alpha_i \in \pi_{q_i}(A)$, $q_i \geq 1$; if no such integer exists W-long $A = \infty$.

We then have the following relationships between these invariants.

Theorem 6.5. Let A be a countable CW-complex then cocat $A \geq$ wcocat $A \geq$ nil $A \geq$ W-long A and furthermore all the inequalities can occur.

This theorem follows from the definitions, Lemma 6.4 of [12] and Theorem 4.6 of [3]. The fact that the inequalities can occur will follow from Examples 8.5, 6.6 and 6.7.

Berstein and Ganea [3; 3.10] show that the space A obtained from the complex projective plane $\mathbb{C}P^2$ by killing off the homotopy groups in dimensions ≥ 6 is not an H-space although nil $A = 1$.

<u>Example 6.6.</u> Let A be defined by the 2-stage Postnikov system

$$K(Z,5) \longrightarrow A \longrightarrow K(Z,2)$$

where the k-invariant is $u^3 \in H^6(Z,2;Z)$, u being the fundamental class in $H^2(Z,2;Z)$. Then wcocat $A = 2$ and nil $A = 1$.

<u>Proof.</u> As we just mentioned Berstein and Ganea proved nil $A = 1$, so it remains to be proved that wcocat $A = 2$.

By ~~Theorem 3~~ Corollary 2 of [28] there is a non-trivial higher order Whitehead product in A, $[\alpha,\alpha,\alpha] = \pm 3!\beta$ where α generates $\pi_2(A) = Z$ and β generates $\pi_5(A) = Z$.

Consider the homotopy sequence of the fibration $D \longrightarrow A \longrightarrow \Omega\Sigma A$,

$$\pi_5(D) \xrightarrow{\partial_*} \pi_5(A) \xrightarrow{e_*} \pi_5(\Omega\Sigma A).$$

By the naturality of the higher order Whitehead product

$$e_*[\alpha,\alpha,\alpha] = [e_*\alpha, e_*\alpha, e_*\alpha] = 0$$

since $\Omega\Sigma A$ is an H-space and all Whitehead products vanish in it. Hence by exactness there exists $\gamma \in \pi_5(D)$ such that $\partial_*(\gamma) = [\alpha,\alpha,\alpha]$ and in particular $\partial_* \neq 0$. Therefore by the definition wcocat $A > 1$.

It follows from ~~Proposition 6.6~~ Lemma 6.3 of [12] that wcocat $A = 2$.

Example 6.7. Let A be defined by the 2-stage Postnikov system

$$K(Z_2,4) \longrightarrow A \longrightarrow K(Z_2,2)$$

where the k-invariant is $u.Sq^1u \in H^5(Z_2,2;Z_2)$, u being the fundamental class in $H^2(Z_2,2;Z_2)$. Then nil $A = 2$ and W-long $A = 1$.

Proof. The space A has trivial homotopy groups except in dimensions 2 and 4. Thus all Whitehead products vanish and W-long $A = 1$.

The k-invariant is non-primative, hence by Theorem 6 of [8] A is not an H-space and cocat $A = 2$. But it also follows from Sugawara's Theorem [31; Theorem 8.1] that ΩA is not homotopy-commutative and hence nil $A = 2$.

In the case of weak category we found two definitions which gave different values on some spaces and we used Ganea's definition to dualize and define weak cocategory. A satisfactory dual of Berstein and Hilton's definition of weak ~~co~~category has not been produced in general. But we could define B-H wcocat $A \leq 1$ if and only if

$\nabla \circ i \simeq 0 : A \flat A \xrightarrow{i} A \vee A \xrightarrow{\nabla} A$. Then as noted in [14] it would follow from Theorem 3.1 of [14] that B-H wcocat $A = 1$ is equivalent to nil $A = 1$.

7. THE COHOMOLOGY RING OF $\Omega\Sigma\mathbb{C}P$

Let $\mathbb{C}P^n$ be the complex projective n-space and $\mathbb{C}P \simeq K(\mathbb{Z},2)$ be the infinite complex projective space. In this chapter we will calculate the cohomology ring of $\Omega\Sigma\mathbb{C}P$ which we will use in Chapter 8. We also show that the same methods can be used to calculate the cohomology ring of $\Omega\Sigma\mathbb{C}P^n$ and ΩS^{n+1}.

Recall the definition of the reduced product complex of a special complex X [21]. For $0 \leqslant k \leqslant \infty$, let $Q_k(X)$ be the disjoint union $\bigcup_{n=0}^{k} X^n$ where $X^0 = *$ and define the relation $\sim$ on $Q_k(X)$ by
$(x_1, x_2, \ldots, x_n) \sim (x_1, x_2, \ldots, \hat{x}_r, \ldots, x_n)$ if $x_r = *$.
Define the reduced product complex $X_k = Q_k(X)/\sim$ and let $\tau: Q_k(X) \longrightarrow X_k$ be the identification map. Let $i: X_k \longrightarrow X_\infty$ be the map which embeds X_k in the first k coordinates of X_∞. It is proved in [21] that there is a weak homotopy equivalence between X_∞ and $\Omega\Sigma X$.

Let X be such that $H^*(X)$ is a free $\mathbb{Z}$-module. In fact in the three cases we will consider $H^*(\mathbb{C}P) = \mathbb{Z}[u]$, $H^*(\mathbb{C}P^r) = \mathbb{Z}[u]/u^{r+1}$ and $H^*(S^n) = \Lambda(u)$. Then

$$H^*(X_k) \approx H^*(X^o) + \sum_{n=1}^{k} \otimes^n \tilde{H}^*(X)$$

and we will denote an element of the n-fold tensor product by $\langle u^{i_1} | u^{i_2} | \ldots | u^{i_n} \rangle$ where $u^{i_r} \in \tilde{H}^*(X)$.

Let $q : X^k \longrightarrow Q_k(X)$ be the natural inclusion and then define a map

$$j : X^k \xrightarrow{\ q\ } Q_k(X) \xrightarrow{\ \tau\ } X_k \xrightarrow{\ i\ } X_\infty .$$

Consider the induced map in cohomology

$$\otimes^k H^*(X) = H^*(X^k) \xleftarrow{\ j^*\ } H^*(X_\infty) = H^*(X^o) + \sum_{n\geq 1} \otimes^n \tilde{H}^*(X).$$

Denote an element of $H^*(X^k) = \otimes^k H^*(X)$ by $x_1^{i_1} x_2^{i_2} \ldots x_k^{i_k}$ where $i_r \geq 0$ and x_r generates $H^*(X)$ in the r-th position of the tensor product.

Then if $n \leq k$,

$$j^* \langle u^{i_1} | u^{i_2} | \ldots | u^{i_n} \rangle = \sum x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n}$$

where the right hand side denotes the sum $\sum x_{\sigma 1}^{i_1}\, x_{\sigma 2}^{i_2} \ldots x_{\sigma n}^{i_n}$ over all order preserving maps $\sigma : \{1,2,\ldots,n\} \longrightarrow \{1,2,\ldots,k\}$. For example, $j^* \langle u | u | \ldots | u \rangle$ is just an elementary symmetric function in $x_1, x_2, \ldots, x_k$.

Now $q^* \circ \tau^*$ is a monomorphism and hence if $k \geq n$ j^* maps the n-fold tensor product monomorphically. We use this fact together with the known ring structure of $H^*(X^k)$ to calculate the ring structure of $H^*(X_k)$

and $H^*(X_\infty)$. If the dimensions of the generators of $H^*(X)$ are even then the cup product of two monomials in $H^*(X^k)$ is just given by ordinary multiplication. Hence the next two theorems follow.

<u>Theorem 7.1.</u> The cohomology ring $H^*((\mathbb{C}P)_k)$ is isomorphic to the ring of functions
$\left\{ \sum' x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \,\middle|\, n \leqslant k \right\}$.

Also $H^*(\Omega \Sigma \mathbb{C}P) = H^*((\mathbb{C}P)_\infty) = \varprojlim_k H^*((\mathbb{C}P)_k)$.

<u>Theorem 7.2.</u> The cohomology ring $H^*((\mathbb{C}P^r)_k)$ is isomorphic to the ring of functions

$\left\{ \sum' x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \,\middle|\, n \leqslant k, x_1^{r+1} = x_2^{r+1} = \cdots = x_n^{r+1} = 0 \right\}$.

Also $H^*(\Omega \Sigma \mathbb{C}P^r) = \varprojlim_k H^*((\mathbb{C}P^r)_k)$.

We can also prove the following well known theorem [30; Proposition 18] in the same way.

<u>Theorem 7.3.</u> $H^*(\Omega S^{n+1})$ has generators $c_r \in H^{rn}(\Omega S^{n+1}) = \mathbb{Z}$ and the ring structure is as follows.

(i) If n is even, $c_p c_q = \dfrac{(p+q)!}{p!\, q!} c_{p+q}$.

(ii) If n is odd $C_pC_q = \begin{cases} 0 \text{ if } p \text{ and } q \text{ are odd} \\ \dfrac{\left[\frac{p+q}{2}\right]!}{\left[\frac{p}{2}\right]!\,\left[\frac{q}{2}\right]!} C_{p+q} \text{ otherwise.} \end{cases}$

<u>Proof.</u> The generators of $H^*(\Omega S^{n+1}) = H^*(S^n_\infty)$ are $\langle u|u|..|u\rangle = C_r \in H\ (S^n_\infty)$ where u is the generator of $H^n(S^n)$. Then $j^*C_r = \sum x_1x_2..x_r$.

(i) If n is even, then $x_ix_j = x_jx_i$ for all i and j. Hence $(\sum x_1)^r = r!\sum x_1x_2..x_r$

so $C_r = \frac{1}{r!}C_1{}^r$ and $H^*(\Omega S^{n+1})$ is an algebra with divided powers. Therefore

$$C_pC_q = \frac{(p+q)!}{p!\ q!} C_{p+q}.$$

(ii) If n is odd then $x_ix_j = -x_jx_i$ for all i and j. Hence $(\sum x_1)^2 = 0$ and so $C_1{}^2 = 0$. Now $(\sum x_1)(\sum x_1x_2..x_{2r}) = \sum x_1x_2..x_{2r+1}$ and so $C_1C_{2r} = C_{2r+1}$. Also

$(x_ix_j)(x_1..\hat{x}_i..\hat{x}_j..x_{2r}) = (-1)^{i+j-1}(x_1..x_{2r})$ and

$$\sum_{i=1}^{2r-1}\sum_{j=i+1}^{2r}(x_ix_j)(x_1..\hat{x}_i..\hat{x}_j..x_{2r}) = \sum_{i=1}^{2r-1}\sum_{j=i+1}^{2r}(-1)^{i+j-1}(x_1..x_{2r})$$

$$= \sum_{\substack{i=1 \\ i \text{ odd}}}^{2r-1}(x_1..x_{2r})$$

$$= r(x_1..x_{2r}).$$

Therefore $(\sum x_1x_2)(\sum x_1..x_{2r-2}) = r(\sum x_1..x_{2r})$ and so $C_2C_{2r-2} = rC_{2r}$. It follows by induction that
$C_{2r} = \frac{1}{r!}\ C_2{}^r$.

Hence if p and q are odd,
$C_pC_q = C_1C_{p-1}.C_1C_{q-1} = 0$ since $C_1{}^2 = 0$.

If p is odd and q even,

$$C_pC_q = C_1 \frac{\left(\frac{p+q-1}{2}\right)!}{\left(\frac{p-1}{2}\right)!\ \left(\frac{q}{2}\right)!}\ C_{p+q-1}$$

$$= \frac{\left[\frac{p+q}{2}\right]!}{\left[\frac{p}{2}\right]!\ \left[\frac{q}{2}\right]!}\ C_{p+q}.$$

We get a similar expression if p is even and q odd and finally if both p and q are even

$$C_pC_q = \frac{\left(\frac{p+q}{2}\right)!}{\left(\frac{p}{2}\right)!\ \left(\frac{q}{2}\right)!}\ C_{p+q}.$$

8. H-SPACES AND WEAK COCATEGORY

In this chapter we find a condition for wcocat $A \leqslant 1$ when A is defined by a Postnikov system with one non-primitive k-invariant. We then find an example of a space A with wcocat $A = 1$ but which is not an H-space.

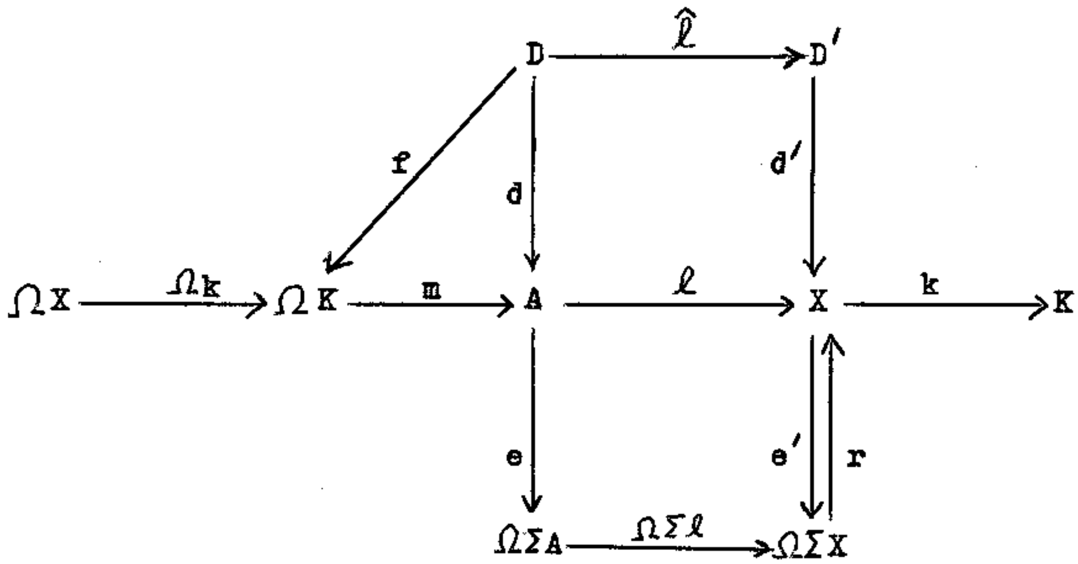
We use this same space to answer a question asked by I. M. James. We show that if in the fibration $\Omega A \xrightarrow{g} A^S \xrightarrow{f} A$ there is a retraction ρ such that $\rho \circ g \simeq 1$ then it does not necessarily follow that A is an H-space.

Throughout this chapter we will consider a space A given by the fibration sequence

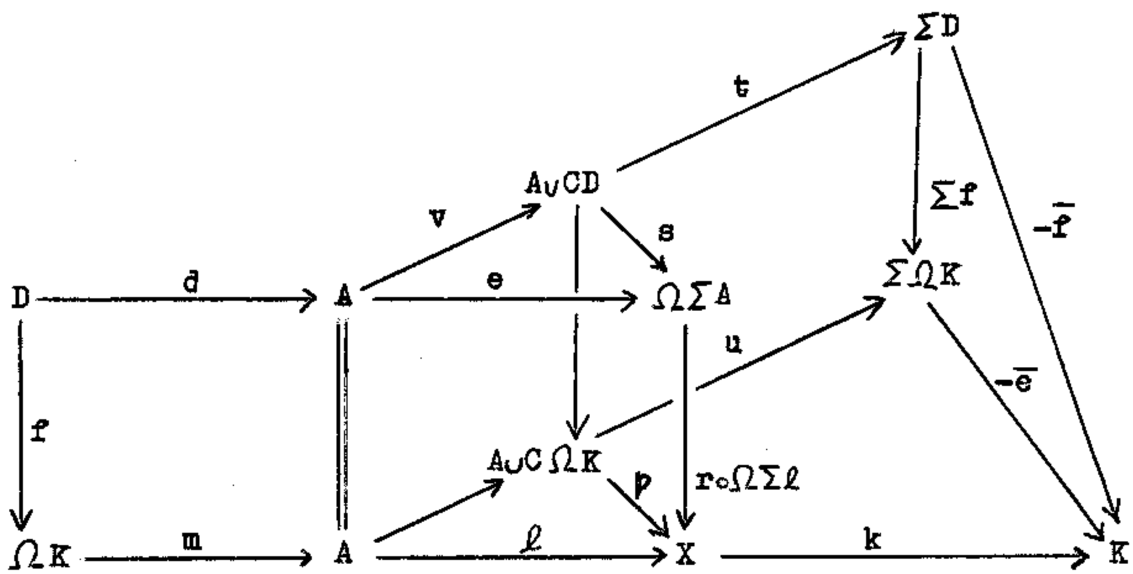
$$\Omega K \xrightarrow{m} A \xrightarrow{\ell} X \xrightarrow{k} K$$

where X is an H-space with multiplication μ, $K = K(\pi, n)$ is an Eilenberg-Maclane space and $k \in H^n(X; \pi)$ is the k-invariant.

Let D and D' be the fibres of the natural embeddings $e: A \longrightarrow \Omega \Sigma A$ and $e': X \longrightarrow \Omega \Sigma X$. The multiplication μ defines a retraction $r: \Omega \Sigma X \longrightarrow X$ [21] such that $r \circ e' \simeq 1$. Hence $d' \simeq 0: D' \longrightarrow X$. Since $\ell \circ d \simeq d' \circ \hat{\ell} \simeq 0$ we can pull d back to a map $f: D \longrightarrow \Omega K$.



The map f is not unique, but we will choose a particular one as follows, obtained from the retraction r.



In the above diagram $r \circ \Omega\Sigma\ell \circ e \simeq r \circ e' \circ \ell \simeq \ell$ and hence this homotopy induces a map $f: D \longrightarrow \Omega K$ between the fibres of e and ℓ. Let $A \cup CD$ be the cofibre of ∂ and from the Puppe sequence there is a map $t: A \cup CD \longrightarrow \Sigma D$. Since $e \circ \partial \simeq 0$ there is a map $CD \longrightarrow \Omega\Sigma A$ which we can use to ~~lift~~ extend e to a map $s: A \cup CD \longrightarrow \Omega\Sigma A$. Also perform the same construction for $\Omega K \longrightarrow A \longrightarrow X$ and obtain the maps $u: A \cup C\Omega K \longrightarrow \Sigma\Omega K$ and $p: A \cup C\Omega K \longrightarrow X$.

Then if $\bar{e}: \Sigma\Omega K \longrightarrow K$ is the evaluation map and $\bar{f}$ is the adjoint of f it will follow from naturality and from Lemma 8.1 that the above diagram is homotopy commutative.

This lemma also follows from Lemma 14 of [26].

__Lemma 8.1.__ $k \circ p \simeq (-\bar{e}) \circ u : A \cup C\Omega K \longrightarrow K$

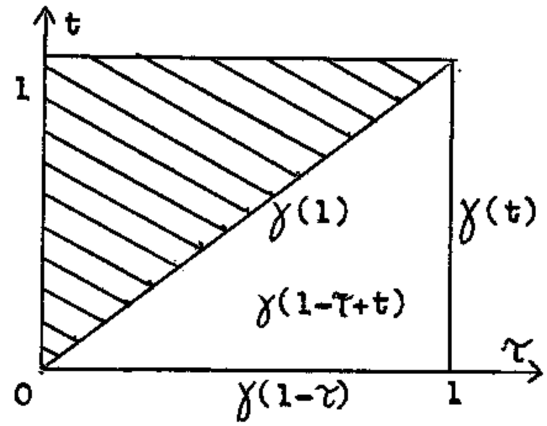
__Proof.__ Let $A = \{(x,\gamma) \in X \times PK \mid \gamma(1) = k(x)\}$ then
$$A \cup C\Omega K = \frac{A \cup \{(x,\gamma) \in C\Omega K\}}{(*,\gamma) \sim (\gamma,1)}.$$

Now $k \circ p(x,\gamma) = k(x) = \gamma(1)$ for $(x,\gamma) \in A$ and $k \circ p(\gamma,\tau) = k(*) = *$ for $(\gamma,\tau) \in C\Omega K$. Also $(-\bar{e}) \circ u(x,\gamma) = -\bar{e}(*) = *$ for $(x,\gamma) \in A$ and $(-\bar{e}) \circ u(\gamma,\tau) = -\bar{e}\langle\gamma,\tau\rangle = \gamma(1-\tau)$ for (γ,τ) $C\Omega K$.

Define a homotopy $H_t: A \cup C\Omega K \longrightarrow K$ by $H_t(x,\gamma) = \gamma(t)$ for $(x,\gamma) \in A$ and

$$H_t(\gamma,\tau) = \begin{cases} \gamma(1-\tau+t) & \text{if } \tau \geq t \\ * & \text{if } \tau < t \end{cases} \quad \text{for } (\gamma,\tau) \in C\Omega K.$$

Then H_t is a well defined homotopy and $H_0 = (-\bar{e})\circ u$ and $H_1 = k \circ p$.



Now consider the definition of the cohomology suspension of the fibration $D \xrightarrow{d} A \xrightarrow{e} \Omega\Sigma A$.

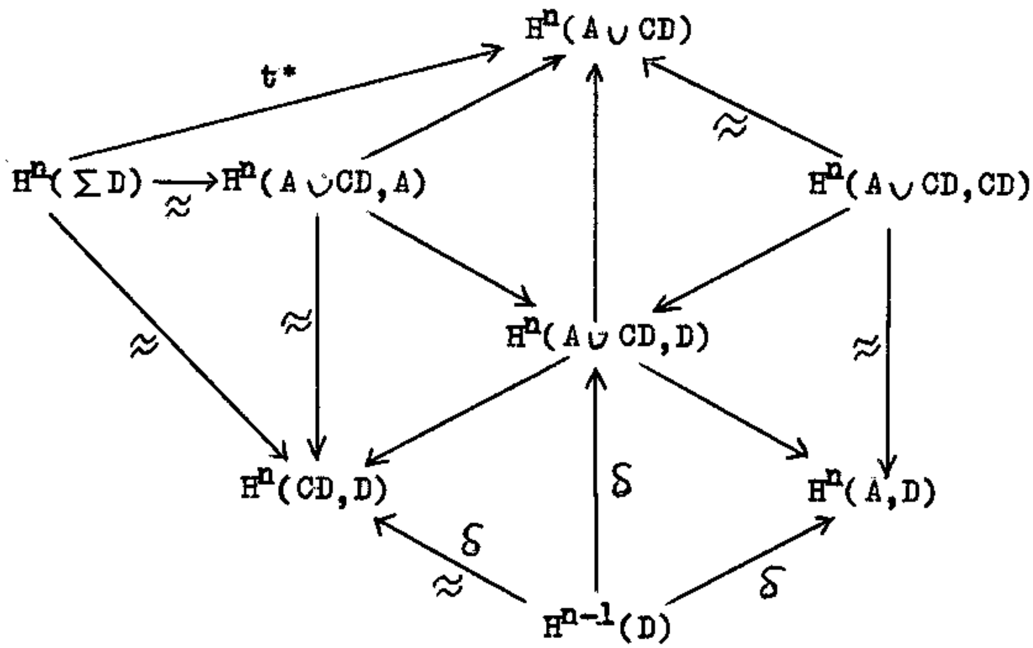
$$\begin{array}{ccccc} H^n(\Omega\Sigma A;\pi) & \xrightarrow{\hat{e}^*} & H^n(A,D;\pi) & \xleftarrow{\delta} & H^{n-1}(D;\pi) \\ \updownarrow & & \updownarrow & & \updownarrow \\ & & H^n(A\cup CD;\pi) & & [D;\Omega K] \\ \updownarrow & & \updownarrow & & \updownarrow \\ [\Omega\Sigma A;K] & \xrightarrow{s^*} & [A\cup CD;K] & \xleftarrow{-t^*} & [\Sigma D;K] \end{array}$$

In the above diagram all the vertical maps are isomorphisms, δ is the boundary homomorphism and

$\hat{e}^*$ is induced from e. The left hand square is commutative by naturality.

<u>Lemma 8.2.</u> The right hand square in the above diagram is commutative.

<u>Proof.</u> Consider the following diagram.



Then by the hexagonal lemma [9; Chapter I, Lemma 15.1] the hexagon is anti-commutative. The two extra triangles on the left are commutative and the result follows by chasing around the outside of the diagram.

The cohomology suspension σ is a homomorphism from a subgroup of $H^n(\Omega\Sigma Y;\pi)$ to a quotient group of $H^n(D;\pi)$:

$$\sigma : \hat{e}^{*-1}(\operatorname{Im}\delta) \longrightarrow \frac{\delta^{-1}(\operatorname{Im} e^*)}{\operatorname{Im} d^*}.$$

<u>Proposition 8.3.</u> $f = \sigma(k\circ r\circ\Omega\Sigma\ell)$.

<u>Proof.</u> We have

$s^*(k\circ r\circ\Omega\Sigma\ell) = k\circ r\circ\Omega\Sigma\ell\circ s$

$= -\bar{f}\circ t$ (by the commutativity of the diagram on p. 55)

$= -t^*(\bar{f})$.

Since the isomorphism between $[D;\Omega K]$ and $[\Sigma D;K]$ is the adjoint isomorphism, it follows from Lemma 8.2 that $\delta(f) = \hat{e}^*(k\circ r\circ\Omega\Sigma\ell)$.

We have the following commutative diagram in which the top line is part of the cohomology exact sequence of a cofibration.

$$\begin{array}{ccccc} H^{n-1}(D;\pi) & \xleftarrow{\ d^*\ } & H^{n-1}(A;\pi) & \xleftarrow{\ v^*\ } & H^{n-1}(A\vee CD;\pi) \\ & & \nwarrow e^* & & s^* \nearrow \\ & & & H^{n-1}(\Omega\Sigma A;\pi) & \end{array}$$

Now e^* is epimorphic and hence so is v^*. Therefore $d^* = 0$ and by the definition of the suspension $f = \sigma(k \circ r \circ \Omega\Sigma\ell)$.

Proposition 8.4. wcocat $A \leqslant 1$ if and only if there exists a map $w: D \longrightarrow \Omega X$ such that

$$\Omega k \circ w = f = \sigma(k \circ r \circ \Omega\Sigma\ell) \in H^{n-1}(D;\pi).$$

This follows from the fact that $d \simeq m \circ f$ and from the homotopy exact sequence of a fibration.

Example 8.5. Let A be defined by the 2-stage Postnikov system

$$K(Z_4,7) \longrightarrow A \longrightarrow K(Z,2)$$

where the k-invariant is $u^4 \in H^8(Z,2;Z_4)$, u being the fundamental class in $H^2(Z,2;Z_4)$. Then cocat $A = 2$ and wcocat $A = 1$.

Proof. There is essentially only one multiplication μ on $K(Z,2)$ and

$$\mu^*(u^4) = 1 \otimes u^4 + 2u^2 \otimes u^2 + u^4 \otimes 1 \quad (\text{mod } 4).$$

Hence k is non-primitive and by [8; Theorem 6] A is not an H-space. Since A is a 2-stage Postnikov system cocat $A = 2$ [12; Lemma 6.3].

Since $\Omega k = 0$, by Proposition 8.4 wcocat $A = 1$

if and only if $f = \sigma((\Omega\Sigma\ell)^*r^*k) = 0 \in H^7(D;Z_4)$.

Now by Theorem 7.1

$$j^*r^*u = \sum x_1 \in H^2(X^k;Z_4).$$

Hence $\quad j^*r^*u^4 = (\sum x_1)^4$

$$= \sum x_1{}^4 + 2\sum x_1{}^2 x_2{}^2 \in H^8(X^k;Z_4).$$

Therefore $\quad r^*k = r^*u^4$

(8.6) $\quad = \langle u^4 \rangle + 2\langle u^2 | u^2 \rangle$.

Now $\ell^*: H^*(X;Z_4) \longrightarrow H^*(A;Z_4)$ is an isomorphism in dimensions less that 7 and a monomorphism in dimension 7 and in dimension 8 $\ell^*k = 0$. In dimensions less that 8 we will use the same notation for the elements of the cohomology of A as those of X.

Hence $(\Omega\Sigma\ell)^*r^*k = \ell^*_\infty r^*k$

$$= \langle \ell^*u^4 \rangle + 2\langle \ell^*u^2 | \ell^*u^2 \rangle$$

$$= 2\langle u^2 | u^2 \rangle.$$

(8.7) By Proposition 8.3 $f = \sigma\, 2\langle u^2 | u^2 \rangle$.

Now consider the cohomology spectral sequence (mod 4) of the fibration

$$D \longrightarrow A \longrightarrow \Omega\Sigma A.$$

We have $E_2{}^{pq} \Longrightarrow Gr\, H^n(A;Z_4)$ where $E_2{}^{pq} = H^p(\Omega\Sigma A; H^q(D;Z_4))$. Since, for $p < 8$, $H^p(\Omega\Sigma A)$ is isomorphic to $H^p(\Omega\Sigma X)$, which is free, it follows that

$$E_2^{pq} \approx H^p(\Omega \Sigma A; Z_4) \otimes H^q(D; Z_4).$$

Now $H^8(A; Z_4)$ is generated by u' where $m^* u' = \beta_4 y$, y being the generator of $H^7(Z_4, 7; Z_4)$ and β_4 is the Bockstein homomorphism associated with the sequence $0 \longrightarrow Z_4 \longrightarrow Z_{16} \longrightarrow Z_4 \longrightarrow 0$.

Generators of E_2^{pq}

q \ p	0	1	2	3	4	5	6	7	8
3	v	0	$\langle u \rangle \otimes v$	0	$\langle u \mid u \rangle \otimes v$ $\langle u^2 \rangle \otimes v$	0			
2	0	0	0	0	0	0	0		
1	0	0	0	0	0	0	0	0	
0	1	0	$\langle u \rangle$	0	$\langle u \mid u \rangle$ $\langle u^2 \rangle$	0	$\langle u \mid u \mid u \rangle$ $\langle u^2 \mid u \rangle$ $\langle u \mid u^2 \rangle$ $\langle u^3 \rangle$	0	$\langle u \mid u \mid u \mid u \rangle$ $\langle u \mid u \mid u^2 \rangle$ $\langle u \mid u^2 \mid u \rangle$ $\langle u^2 \mid u \mid u \rangle$ $\langle u \mid u^3 \rangle \langle u^3 \mid u \rangle$ u' $\langle u^2 \mid u^2 \rangle$

In this spectral sequence all the elements are eventually killed off except $H^n(A; Z_4) \subset E^{n,0}$. Hence there exists $v \in E_2^{0,3}$ such that $d_4(v) = \langle u \mid u \rangle$.

Then $\quad \partial_4(\langle u|u\rangle \otimes v) = \langle u|u\rangle\langle u|u\rangle$. But by Theorem 7.1 $\quad j^*(\langle u|u\rangle\langle u|u\rangle) = (\sum x_1x_2)^2$

$$= \sum x_1^2x_2^2 + 2\sum x_1x_2x_3^2$$
$$+ 2\sum x_1x_2^2x_3 + 2\sum x_1^2x_2x_3 + 6\sum x_1x_2x_3x_4.$$

Hence $\partial_4(\langle u|u\rangle \otimes v) = \langle u^2|u^2\rangle + 2\{\langle u|u|u^2\rangle + \langle u|u^2|u\rangle + \langle u^2|u|u\rangle + \langle u|u|u|u\rangle\}$.

So in $E_8^{8,0}$

$$\langle u^2|u^2\rangle = 2\{\langle u|u|u^2\rangle + \langle u|u^2|u\rangle + \langle u^2|u|u\rangle + \langle u|u|u|u\rangle\}.$$

Therefore $2\langle u^2|u^2\rangle = 0 \pmod 4$ in $E_8^{8,0}$ and so $2\langle u^2|u^2\rangle$ does not belong to the image of the cohomology transgression.

Hence by (8.7) $f = 0 \in H^7(D;Z_4)$ and $d \simeq m\circ f \simeq 0$ and so wcocat $A = 1$.

This completes the proof of Example 8.5.

For any space A there is a fibration $\mathcal{Q}: \Omega A \xrightarrow{g} A^S \xrightarrow{f} A$ where A^S is the space of free loops in A and if $\lambda \in A^S$ is a free loop then $f(\lambda) = \lambda(0)$, the image of the base point of the loop. Then we have the following known theorem about the fibration $\mathcal{Q}$. (James, I. M and Thomas, E; An approach to the Enumeration Problem for Non-stable Vector Bundles; Th. 2.7)

<u>Proposition 8.8.</u> If A is an H-space and also a countable CW-complex, then $\mathcal{Q}$ has a retraction $\rho: A^S \longrightarrow \Omega A$ such that $\rho\circ g \simeq 1$.

Proof. Let μ be a multiplication on A such that $\mu(a,*) = \mu(*,a) = a$ for all $a \in A$. This is possible by Lemma 6.4 of [23]. Then by Theorem 1.1 of [22] there exists a map $\nu : A \longrightarrow A$ such that $\mu\circ(\nu \times 1)\circ\Delta \simeq 0 : A \xrightarrow{\Delta} A\times A \xrightarrow{\nu\times 1} A\times A \xrightarrow{\mu} A$. Let $H_s : A \longrightarrow A$ be a homotopy in which $H_0 = 0$ and $H_1 = \mu\circ(\nu\times 1)\circ\Delta$.

Define the retraction $\rho : A^S \longrightarrow \Omega A$ by

$$(\rho,\lambda)(t) = \begin{cases} H_{3t}(\alpha) & \text{if } 0 \leqslant t \leqslant 1/3 \\ \mu(\nu\lambda(0), \lambda(3t-1)) & \text{if } 1/3 \leqslant t \leqslant 2/3 \\ H_{3-3t}(\alpha) & \text{if } 2/3 \leqslant t \leqslant 1 \end{cases}$$

where $\alpha = \mu(\nu\lambda(0), \lambda(0)) \in A$.

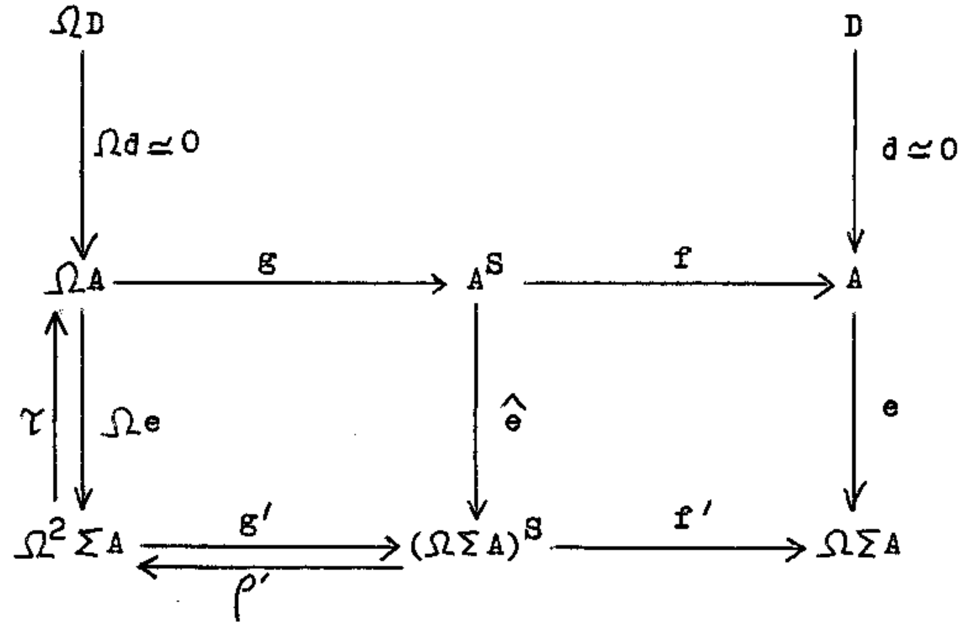
Then
$$\rho\circ g\circ\lambda(t) = \begin{cases} * & \text{if } 0 \leqslant t \leqslant 1/3 \\ \lambda(3t-1) & \text{if } 1/3 \leqslant t \leqslant 2/3 \\ * & \text{if } 2/3 \leqslant t \leqslant 1 \end{cases}$$

and hence $\rho\circ g \simeq 1$.

T. Ganea has pointed out that Example 8.5 can be used to show that the converse to Proposition 8.8 is not true.

Proposition 8.9. (Ganea) Let wcocat $A \leqslant 1$ then the fibration $\mathcal{O}\!\!\!\alpha : \Omega A \xrightarrow{g} A^S \xrightarrow{f} A$ has a retraction ρ such that $\rho\circ g \simeq 1$.

<u>Proof.</u> Consider the following commutative diagram.



Since $\Omega \Sigma A$ is an H-space, by Proposition 8.8 g' has a retraction ρ'. Since wcocat $A \leq 1$, $d \simeq 0$ and it follows from the fibration sequence that Ωe has a retraction $\tau : \Omega^2 \Sigma A \longrightarrow \Omega A$ such that $\tau \circ \Omega e \simeq 1$ [20; Lemma 2.3].

Define $\rho : A^S \longrightarrow \Omega A$ by $\rho = \tau \circ \rho' \circ \hat{e}$.

Then
$$\begin{aligned} \rho \circ g &\simeq \tau \circ \rho' \circ \hat{e} \circ g \\ &\simeq \tau \circ \rho' \circ g' \circ \Omega e \\ &\simeq \tau \circ \Omega e \\ &\simeq 1. \end{aligned}$$

The following example now comes from Example 8.5 and Proposition 8.9.

<u>Example 8.10.</u> Let A be the space defined in Example 8.5, then the fibration $\mathcal{P}: \Omega A \longrightarrow A^S \longrightarrow A$ has a retraction, but A is not an H-space.

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CORRECTIONS

P 46 l 10. Change 'Theorem 3' to 'Corollary 2'.

P 47 l 1. Change 'Proposition 6.6' to 'Lemma 6.3'.

P 51 l 5. Theorem 7.1. The additive generators of $H^*((CP)_k)$ are isomorphic to $\left\{\sum x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n} \,\middle|\, n \leqslant k\right\}$ and the ring structure is given by the standard multiplication of polynomials.

Also $H^*(\Omega\Sigma CP) = H^*((CP)_\infty) = \varprojlim_k H^*((CP)_k)$.

Theorem 7.2. The additive generators of $H^*((CP^r)_k)$ are isomorphic to $\left\{\sum x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n} \,\middle|\, n \leqslant k, x_1^{r+1} = x_2^{r+1} = \ldots = x_n^{r+1} = 0\right\}$ and the ring structure is given by the standard multiplication of polynomials.

Also $H^*(\Omega\Sigma CP^r) = \varprojlim_k H^*((CP^r)_k)$.

P 56 l 6. Change 'lift' to 'extend'.

P 57 l -3. In the diagram change $[\Omega\Sigma Y;K]$ to $[\Omega\Sigma A;K]$.

P 63 l 13. Change 'wcocat Y' to 'wcocat A'.

l -4. Add: (James, I.M. and Thomas, E.; An approach to the ennumeration problem for non-stable vector bundles; J. Math. Mech. 14 (1965) pp.485-506; Theorem 2.7).

P 69 l 13. Higher order Whitehead products and Postnikov systems; Illinois J. Math. 11 (1967) pp.414-416.