Geometry of Radix Representations

William J. Gilbert*

1. Introduction

The aim of this paper is to illuminate the connection between the geometry and the arithmetic of the radix representations of the complex numbers and other algebraic number fields. We indicate how these representations yield a variety of naturally defined fractal curves and surfaces of higher dimensions.

As is well known, the natural numbers can all be represented using any integer b, larger than one, as base, with the digits $0, 1, 2, \ldots, b-1$. All the integers, both positive and negative, can be represented without signs by means of the negative integral base b, less than minus one, using the natural numbers $0, 1, 2, \ldots, |b|-1$ as digits [6, §4.1]. Each Gaussian integer may be uniquely represented in binary form as $\sum_{k=0}^{r} a_k(-1+i)^k$, where each $a_k = 0$ or 1, [1, §4.3; 6, §4.1]. We will unify and generalize such representations.

2. Algebraic Number Fields

We now describe more precisely what we mean by a radix representation in an algebraic number field. Let ρ be an algebraic integer whose minimum polynomial is $x^n + p_{n-1}x^{n-1} + \cdots + p_1x + p_0$; *let*

$$N = |\mathrm{Norm}(\rho)| = |(-1)^n p_0|.$$

We will try to represent elements of the algebraic number field $\mathbb{Q}(\rho)$ using the radix ρ and natural numbers as digits. We restrict ourselves here to only considering digits which are natural numbers, as this appears to be the obvious

*Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1.

generalization of the familiar number systems and it is more convenient for doing arithmetical calculations. However, usually for geometric reasons, it is sometimes necessary to use nonintegral digits. In such cases, the results obtained may be slightly different. The largest set of algebraic numbers we could expect to represent, without using negative powers of the radix, is the ring $\mathbb{Z}[\rho]$. Note that this ring may not be the whole ring of algebraic integers $\mathbb{A} \cap \mathbb{Q}(\rho)$ in the number field. We say that ρ is the base (or radix) of a *full radix representation* of $\mathbb{Z}[\rho]$ if each element z of $\mathbb{Z}[\rho]$ can be written in the form $z = \sum_{k=0}^{r} a_k \rho^k$, where the digits a_k are natural numbers such that $0 \leqslant a_k < N$. We denote this representation by $z = (a_r a_{r-1} \ldots a_1 a_0)_\rho$.

The reason that the norm yields the correct number of digits is due to the following observation.

Lemma . *Let c and d be two integers in $\mathbb{Z}$. Then $c \equiv d \pmod{\rho}$ in $\mathbb{Z}[\rho]$ if and only if $c \equiv d \pmod{N}$ in $\mathbb{Z}$.*

Proof. Suppose $c \equiv d \pmod{\rho}$ in $\mathbb{Z}[\rho]$. Then there exist rational integers q_i such that

$$\begin{aligned} c - d &= \rho(q_n\rho^{n-1} + \cdots + q_2\rho + q_1) = q_n\rho^n + \cdots + q_2\rho^2 + q_1\rho \\ &= -q_n(p_{n-1}\rho^{n-1} + \cdots + p_1\rho + p_0) + q_{n-1}\rho^{n-1} + \cdots + q_2\rho^2 + q_1\rho \\ &= (q_{n-1} - q_n p_{n-1})\rho^{n-1} + \cdots + (q_1 - q_n p_1)\rho - q_n p_0. \end{aligned}$$

Since $1, \rho, \rho^2, \ldots, \rho^{n-1}$ are linearly independent over $\mathbb{Q}$, it follows that $c - d = -q_n p_0$. As $N = |p_0|$, we have $c \equiv d \pmod{N}$.

Now $N = \pm p_0 = \mp\rho(\rho^{n-1} + p_{n-1}\rho^{n-2} + \cdots + p_1)$, so that N is divisible by ρ in $\mathbb{Z}[\rho]$ and the converse implication follows. □

This lemma implies that the quotient ring $\mathbb{Z}[\rho]/(\rho)$ is isomorphic to $\mathbb{Z}_N$ and that $0, 1, 2, \ldots, N-1$ form a complete set of representatives of the congruence classes modulo ρ in $\mathbb{Z}[\rho]$. Clearly, the digits of any radix representation of $\mathbb{Z}[\rho]$ must form a complete set of representatives of these classes.

If an element of $\mathbb{Z}[\rho]$ can be represented using the base ρ and digits $0, 1, 2, \ldots, N-1$, the representation is unique. It does not matter whether ρ yields a full or only a partial radix representation. The proof of the uniqueness uses the above lemma and is the same as for ordinary decimals.

Katai and Szabo [5] show that, for each positive integer m, the Gaussian integers can be represented by the radix $-m + i$ (and $-m - i$) using the digits $0, 1, 2, \ldots, m^2$. In particular, the complex numbers can be written as "decimals" in base $-3 + i$; for example, $(241)_{-3+i} = 2(-3 + i)^2 + 4(-3 + i) + 1 = 5 - 8i$. The bases mentioned above are the only ones that will represent all the Gaussian integers in the required form. Knuth (see [6, §4.1]) has defined a "quater-imaginary" number system for the complex numbers based on the radix $2i$, which has norm 4. All the elements of $\mathbb{Z}[2i]$, that is, Gaussian integers with even imaginary parts, can be uniquely represented in this system. Gaussian integers with odd imaginary parts can be represented if we allow expansions to one radix place; for example $(31.2)_{2i} = 3(2i) + 1 + 2(2i)^{-1} = 1 + 5i$.

For the complex quadratic fields $\mathbb{Q}(\sqrt{-m})$, where $-m \equiv 2, 3 \pmod 4$ and $-m \neq -1$, one good base is provided by $\sqrt{-m}$ itself. Given any integer $a + b\sqrt{-m}$ in the field, first write the rational integers $a = (a_r \ldots a_1 a_0)_{-m}$ and $b = (b_s \ldots b_1 b_0)_{-m}$ in base $-m$. It then follows that

$$a + b\sqrt{-m} = (b_t a_t b_{t-1} a_{t-1} \ldots b_1 a_1 b_0 a_0)_{\sqrt{-m}},$$

where $t = \max(r, s)$.

Given an arbitrary number field, it is not always possible to find a base for its integers. For example, in the biquadratic field $\mathbb{Q}(\sqrt{7}, \sqrt{10})$ there is no integer α such that $\mathbb{Z}[\alpha] = \mathbb{A} \cap \mathbb{Q}(\sqrt{7}, \sqrt{10})$ (see [8, p. 46]). However, it may still be possible to represent the integers in the field by allowing radix expansions using negative powers of the base.

The usual arithmetic operations of addition and multiplication can be performed using these radix representations in much the same way as ordinary arithmetic base N. The only difference is in the carry digits. For example, the root ρ of the cubic $P(x) = x^3 + x^2 + x + 2$ is a base for $\mathbb{Z}[\rho]$. Since ρ is also a root of $(x-1)P(x)$, we have $\rho^4 + \rho = 2$ and so $2 = (10010)_\rho$. Hence, whenever we have an overflow of 2 in any one column when doing an arithmetical operation, we have to carry 1001 to the next four higher columns.

3. Geometry of Representations

The elements of $\mathbb{Q}(\rho)$ can be pictured as points in $\mathbb{Q}^n$ using coordinates $1, \rho, \rho^2, \ldots, \rho^{n-1}$. However, if $\mathbb{Q}(\rho) = \mathbb{Q}(i)$ it is often more useful to use the Argand diagram instead. In $\mathbb{Q}^n$, the points of $\mathbb{Z}[\rho]$ correspond to the integer lattice points. The radix representations in base ρ map injectively to the lattice points. If ρ is a base for a full representation of $\mathbb{Z}[\rho]$, then all the lattice points will be covered; if not, the image will be some infinite subset.

These images can be viewed as n-dimensional jigsaw puzzles whose rth piece consists of the union of unit n-dimensional cubes centered at the points whose base ρ representation is of length r. The $(r+1)$st piece is formed from $N-1$ copies of the first r pieces translated in $\mathbb{Q}^n$ along the directions of ρ^r, $2\rho^r, \ldots, (N-2)\rho^r$, and $(N-1)\rho^r$. For example, in the jigsaw in the Argand diagram in Figure 1 derived from the base $1 - i$, each piece is twice the size of the previous piece. Each little square corresponds to one Gaussian integer with the origin at the center black square. Since the jigsaw only fills up half the Argand diagram, $1 - i$ only provides a partial radix representation of the Gaussian integers. However, exactly the same pieces put together in Figure 2 using base $-1 + i$ fill the entire plane; this demonstrates the fact that $-1 + i$ is a base for all the Gaussian integers.

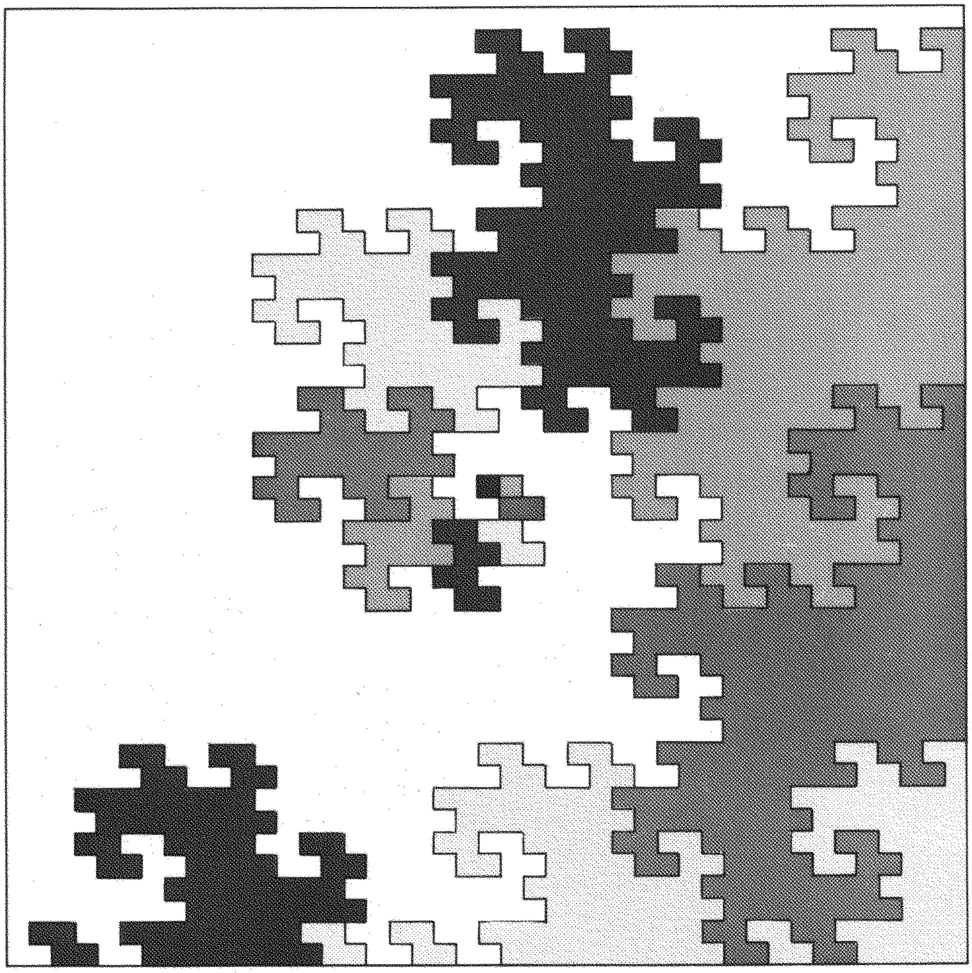

Figure 1. The Gaussian integers in base $1 - i$.

In Figure 3, the elements of $\mathbb{Z}[\omega]$ are represented in the base $-2 - \omega$, where ω is a complex cube root of unity. This base is a root of $x^2 + 3x + 3$, so it has norm 3. Each element of $\mathbb{Z}[\omega]$ is pictured as a unit hexagon in the Argand diagram with the origin being the black one. The figure shows the radix representation up to six places, and if continued it would fill the plane, since $-2 - \omega$ is a good base for $\mathbb{Z}[\omega]$.

Figures 4 and 5 show three-dimensional models derived from the cubic fields generated by the polynomials $x^3 + x^2 + x - 2$ and $x^3 + x^2 + x + 2$ respectively. The former only yields a partial representation, while the latter, if extended, would fill the whole of $\mathbb{Z}^3$ and so provide a full radix representation.

C. Davis and D. Knuth [2] use bases $1 + i$ and $1 + 2\omega$ in their investigation of the dragon and ter-dragon curves in the Argand diagram.

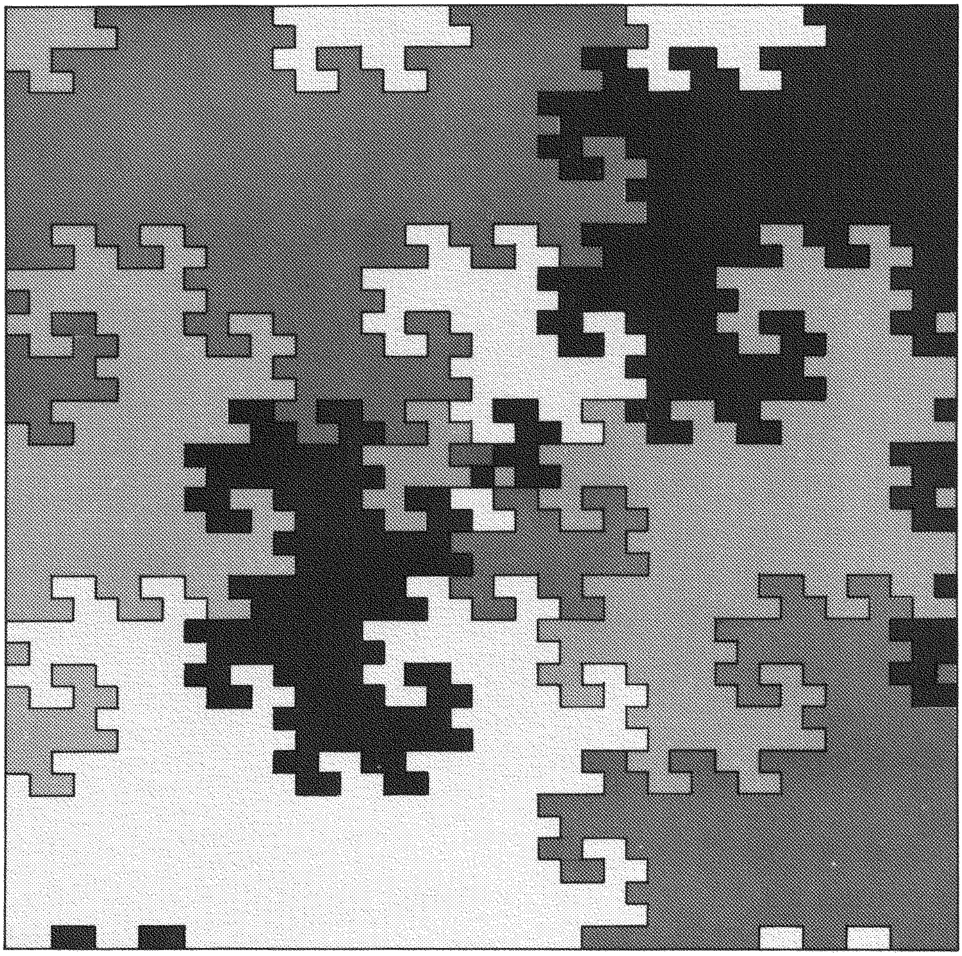

Figure 2. The Gaussian integers in base $-1 + i$.

4. Fractal Curves and Surfaces

It is natural to extend a radix representation to an infinite expansion using negative powers of the base. We say that an element of $\mathbb{Q}(\rho)$ can be written in base ρ if it has an expansion of the form $\sum_{k=-\infty}^{r} a_k \rho^k$ where $0 \leqslant a_k < N$ for all k; we denote this expansion by $(a_r a_{r-1} \ldots a_0 \cdot a_{-1} a_{-2} \ldots)_\rho$. Terminating expansions correspond to elements of $\mathbb{Q}(\rho)$ whose denominators are some power of the norm.

From a geometric point of view, it is tempting to try to complete the representations of $\mathbb{Q}^n$ to representations of $\mathbb{R}^n$. However, if $1, \rho, \rho^2, \ldots, \rho^{n-1}$ are linearly dependent over $\mathbb{R}$, different points of $\mathbb{R}^n$ would correspond to the same number. Therefore, besides the rational numbers, the only fields whose represen-

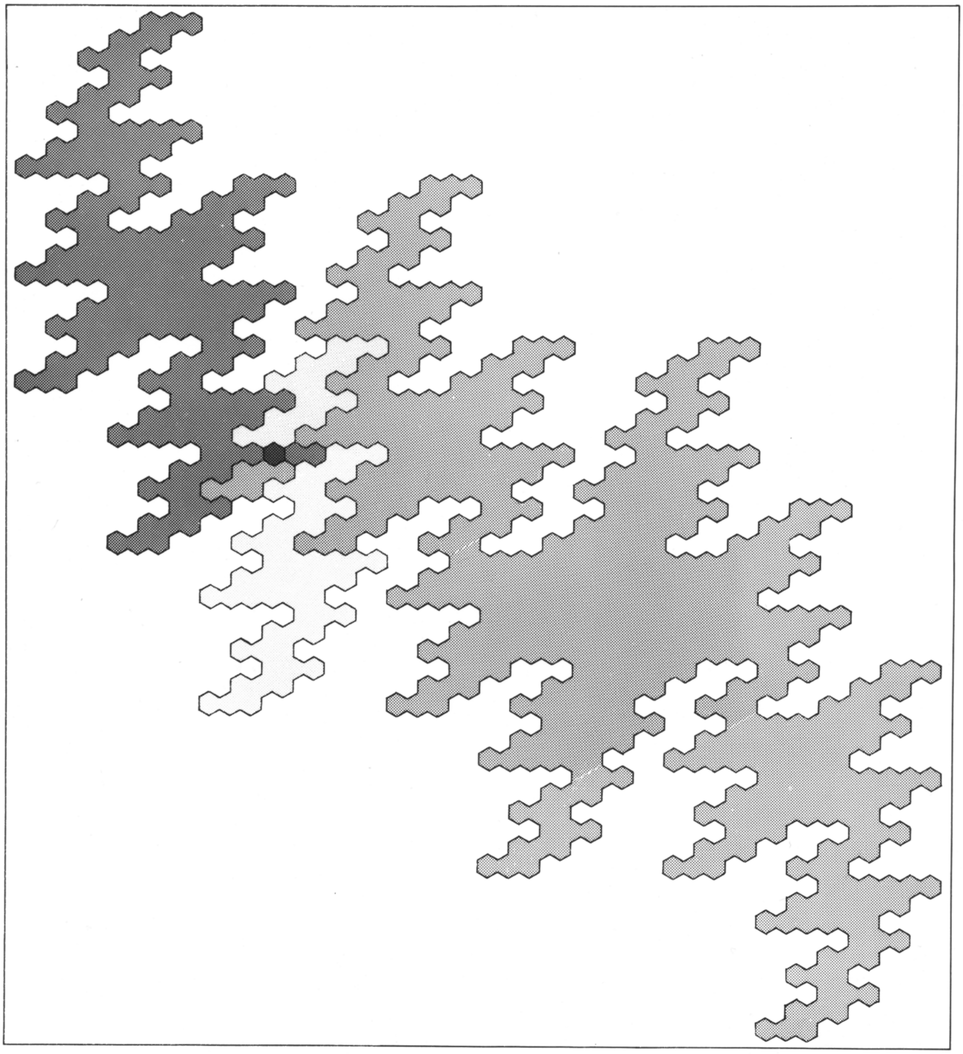

Figure 3. A fragment of the elements of $\mathbb{Z}[\omega]$ in base $-2-\omega$.

tations we could complete are the complex quadratic fields; these fields can all be represented in the Argand diagram.

We find that these complex quadratic fields yield some fascinating geometry by examining the regions of the Argand diagram corresponding to radix expansions of a given form. The regions whose points have expansions of the form $(a_r \ldots a_0 \cdot a_{-1} \ldots)_\rho$, for some fixed power r, have boundaries that are naturally defined fractal curves. Figure 6 shows all the complex numbers that are representable in base $1-i$ using expansions of any length. This region is in fact two space-filling dragon curves joined tail to tail. Mandelbrot [7, p. 313] has calculated the fractal (i.e. Hausdorff) dimension of the dragon's "skin," and it is approximately 1.5236.

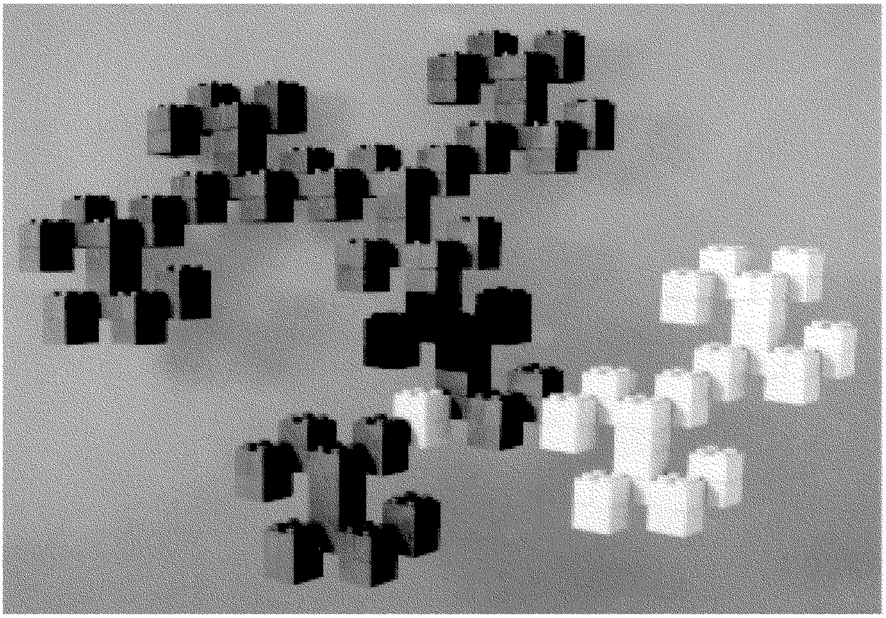

Figure 4. The elements of $\mathbb{Z}[\rho]$ in base ρ where $\rho^3 + \rho^2 + \rho - 2 = 0$.

Figure 7 is a close-up of the Argand diagram in which each region consists of numbers having a fixed integer part in base $-1 + i$. (The axes in this figure are at 45° to the edges.) The boundaries have the same fractal dimension as that of Figure 6. Points on the boundary of two regions will have two representations in base $-1 + i$; each have different integral parts. Since the Argand diagram is two-dimensional, there must be some points that lie on the boundary of three regions, and they have three different representations; for example, $(2 + i)/5 = (0.\overline{011})_{-1+i} = (1.\overline{110})_{-i+1} = (1110.\overline{101})_{-1+i}$, where the bars over the digits indicate that they are to be repeated indefinitely.

For each base $-m + i$ of the complex numbers, we can show [3] that the fractal dimension of the boundary of the resulting regions is

$$(\log\lambda_m)/\log\sqrt{m^2+1}\ ,$$

where λ_m is the positive root of $\lambda^3 - (2m - 1)\lambda^2 - (m - 1)^2\lambda - (m^2 + 1)$.

For an arbitrary number field $\mathbb{Q}(\rho)$, the boundary of the resulting regions in $\mathbb{Q}^n$ may not contain as many points as we desire, because $\mathbb{Q}^n$ is not complete. However, we can still define the fractal dimension of the edge of a region S in $\mathbb{Q}^n$ as follows. Let $\epsilon > 0$ and let E_ϵ be the set of points within ϵ of the edge of S, that is, points whose ϵ-neighborhood contains points of S and points not in S. For each positive number d, cover E by balls of radius $\sigma_i < 2\epsilon$ and take the following infimum over all such coverings:

$$m_d^\epsilon = \inf\sum_i \sigma_i^d.$$

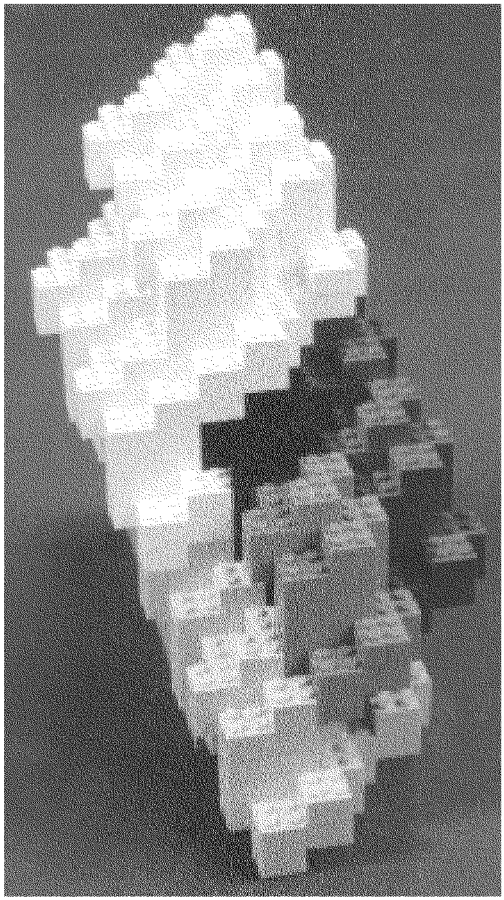

Figure 5. A fragment of the elements of $\mathbb{Z}[\rho]$ in base ρ where $\rho^3 + \rho^2 + \rho + 2 = 0$.

Now let $m_d = \sup_{\varepsilon > 0} m_d^\varepsilon$; this number is proportional to the d-dimensional measure of the edge. The edge is said to have *fractal dimension* D if

$$m_d = \begin{cases} \infty & \text{for all } d < D, \\ 0 & \text{for all } d > D. \end{cases}$$

This fractal dimension is a metric invariant and hence will remain unchanged under a linear transformation. Therefore, whether we represent a complex quadratic field by points in $\mathbb{Q}^2$ or by points in the Argand diagram, we will obtain the same dimension.

Some bases ρ which only yield partial radix representations of $\mathbb{Z}[\rho]$ may not give any infinite convergent radix expansions. For example, all infinite radix expansions using the base of Figure 4 diverge because one of the roots of the minimum polynomial, $x^3 + x^2 + x - 2$, has modulus smaller than one. Therefore fractal surfaces cannot be constructed from this base. On the other hand, the periodic radix expansions using the base ρ of Figure 5 do converge to points of

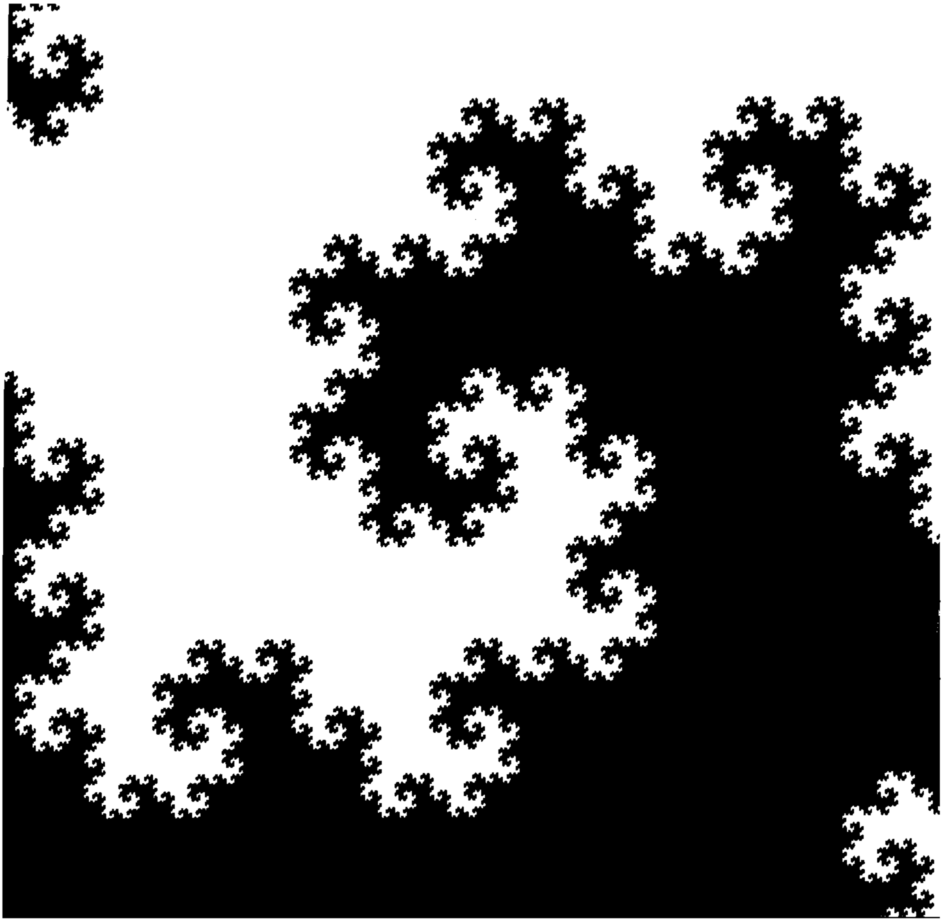

Figure 6. All the complex numbers representable in base $1 - i$.

$\mathbb{Q}(\rho)$, and this base will yield a fractal surface of dimension between two and three.

5. Problems

These radix representations suggest many interesting problems, both geometric and arithmetic. We mention three here.

Firstly, which algebraic integers yield full radix representations? For the quadratic fields we can show that a root of the irreducible polynomial $x^2 + cx + d$ gives a full radix representation if and only if $d \geqslant 2$ and $-1 \leqslant c \leqslant d$. A root of the linear polynomial $x + d$ yields a complete representation if and only $d \geqslant 2$.

Secondly, find an algorithm for dividing a number in base ρ by a rational

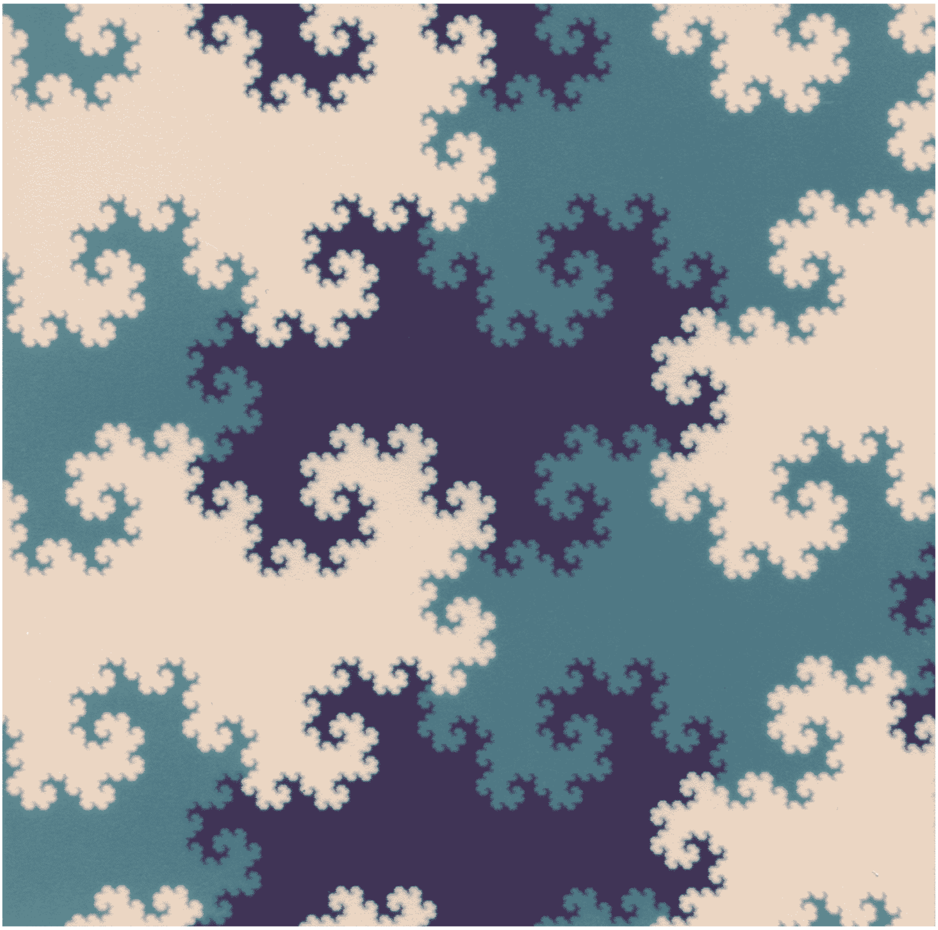

Figure 7. Complex numbers with given integer parts in base $-1+i$. (Thanks are due to John Beatty, who programmed this at Lawrence Livermore Lab.)

integer. In [4] we give such an algorithm in the case of the negative integral bases.

Thirdly, calculate the fractal dimensions of the edges of the regions derived from the representation whose base is the root of a given polynomial. In occasional cases, such as $\rho = \sqrt{-m}$, this dimension will be integral, but it seems that most bases yield fractal curves or surfaces of nonintegral dimension.

References

[1] Akushskiĭ, I. Ia., Amerbaev, V. M., and Pak, I. T., *Osnovy Mashinnoĭ Arifmetiki Kompleksnykh Chisel.* Nauka, Alma-Ata, Kazakhstan SSR 1970.

[2] Davis, Chandler and Knuth, Donald E., Number representations and dragon curves—I, II. *J. Recreational Math.* **3** (1970), 66–81, 133–149.

[3] Gilbert, William J., The fractal dimension of snowflake spirals, *Notices Amer. Math. Soc.* **25** (1978), A-641.

[4] Gilbert, William J. and Green, R. James, Negative based number systems, *Math. Mag.* **52** (1979), 240–244.

[5] Katai, I. and Szabo, J., Canonical number systems for complex integers, *Acta Sci. Math. (Szeged)* **37** (1975), 255–260.

[6] Knuth, Donald E., *The Art of Computer Programming, Vol. 2, Seminumerical Algorithms*. Addison-Wesley, Reading, Mass. 1969.

[7] Mandelbrot, Benoit B., *Fractals; Form, Chance and Dimension*. Freeman, San Francisco 1977.

[8] Marcus, Daniel A., *Number Fields*. Springer-Verlag, New York 1977.