John E. Freund
Arizona State University
Tempe

Richard J. Friedlander
University of Missouri
St. Louis

Bernard A. Fusaro
Salisbury State University
Salisbury, Maryland

David Fuys
Brooklyn College
Brooklyn, New York

Stavroula K. Gailey
Christopher Newport University
Newport News, Virginia

Joseph Gani
Australian National University
Canberra

Robert A. Garden
University of British Columbia
Vancouver, British Columbia, Canada

Joan B. Garfield
University of Minnesota
Minneapolis

Lynn E. Garner
Brigham Young University
Provo, Utah

June L. Gastón
Borough of Manhattan Community College
New York

M. Katherine Gavin
University of Connecticut
Storrs

Dorothy Geddes (deceased)
Brooklyn College
Brooklyn, New York

William, J. Gilbert
University of Waterloo
Waterloo, Ontario, Canada

Alice J. Gill
American Federation of Teachers
Washington, DC

Lynda B. Ginsburg
University of Pennsylvania
Philadelphia

Terry Goodman
Central Missouri State University
Warrensburg

Ivor Grattan-Guinness
Middlesex University
Enfield, Middlesex, England

Mary W. Gray
American University
Washington, DC

Virginia Gray
Columbia College
Sonora, California

Judy Green
Marymount University
Arlington, Virginia

Louise S. Grinstein
Kingsborough Community College
Brooklyn, New York

Douglas A. Grouws
University of Iowa
Iowa City

Rod Haggarty
Oxford Brookes University
Headington, Oxford, England

Bettye C. Hall
BCH Consultants
Houston, Texas

Kathleen M. Harmeyer
ExperTech Corporation
Timonium, Maryland

Mary M. Hatfield
Arizona State University
Tempe

Anne Hawkins
International Association for Statistical Education
London, England

William A. Hawkins, Jr.
Mathematical Association of America
Washington, DC

Jerome D. Hayden
McClean County Unit District 5
Normal, Illinois

Christoper C. Healy
El Monte School District
California

BASES, COMPLEX

Complex numbers in terms of which other complex numbers may be expressed. Complex numbers can be written as single numbers in positional notation using various complex bases, without separating the numbers into their real and imaginary parts. These representations yield some interesting fractal patterns in the complex plane.

A complex integer (or Gaussian integer) is a complex number $x + iy$, in which x and y are real integers. The complex integer z is said to be represented in the complex integer base b, using a digit set D, if it can be written as $z = a_m b^m + a_{m-1} b^{m-1} + \cdots + a_2 b^2 + a_1 b + a_0$, where $a_m, a_{m-1} \ldots a_2, a_1, a_0$ are digits in the set D. This representation is denoted by the positional notation $(a_m a_{m-1} \ldots a_2 a_1 a_0)_b$. For example, all complex integers can be represented uniquely in the base $-1 + i$ using the binary digit set $\{0,1\}$. In this base, $-4 - i$ is represented by $(10111)_{-1+i}$ since $(-1 + i)^4 + (-1 + i)^2 + (-1 + i) + 1 = -4 = i$, and 2 is represented by $(1100)_{-1+i}$ since $(-1 + i)^3 + (-1 + i)^2 = 2$. The complex integer $p_0 + iq_0$ can be converted to the base $-1 + i$ as follows. The right digit, a_0, is 0 if $p_0 + q_0$ is even, and 1 otherwise. Then calculate

$$p_1 + iq_1 = \frac{p_0 + iq_0}{-1 + i} - a_0$$

The next digit, a_1, is 0 if $p_1 + q_1$ is even, and 1 otherwise. Continue in this manner until $p_{m+1} + q_{m+1}$ is zero.

A good exercise for students in complex arithmetic is to have them determine which complex integers can be represented in the base $1 + i$ using the binary digit set $\{0, 1\}$ and expansions of length less than or equal to m. Then have them do the same exercise with the base $-1 + i$. This can lead to the construction, by hand or by computer, of various fractal sets in the complex plane (see Gilbert 1982). Let each square of a sheet of graph paper (or pixel on a computer screen) correspond to a complex integer. If that complex integer can be represented in the base, color the square according to the length of the representation. One fractal obtained this way is called the *space-filling twin dragon curve.* The complex base $-1 + i$ can be used to analyze various dragon curves, which are obtained by repeatedly folding a strip of paper in half and then opening it up; see Davis and Knuth (1970) and Gardner (1990).

The first examples of complex numbers as bases were given in the 1960s; see Knuth (1973, Ch. 4.1) for the history of positional number systems. Only certain bases and digit sets can be used to represent all the complex integers. For each positive integer n, $b = -n + i$ can be used as a base for all the complex integers, using the natural number digit set $D = \{0,1,2, \ldots, n^2\}$. Hence the base $-3 + i$ yields a decimal representation of all the complex integers, using the digits 0, 1, . . . , 9, where, for example, $20 - 159i = (21073)_{-3+i}$. The digits could be complex; one interesting system is to use the base $2 + i$, with the symmetrical digit set $\{0, 1, -1, i, -i\}$.

If all the complex integers can be represented in a given base, then the noninteger complex numbers can also be represented, using expansions to the right of the radix point, in the form $(a_m \ldots a_1 a_0 . a_{-1} a_{-2} \ldots)_b = a_m b^m + \cdots + a_1 b + a_0 + a_{-1} b^{-1} + a_{-2} b^{-2} + \ldots$ where $a_m, \ldots, a_1, a_0, a_{-1}, a_{-2}, \ldots$ are digits in the set D. The usual arithmetic operations of addition, subtraction, and multiplication can be generalized, with some interesting twists, to complex bases; see Gilbert (1984). For example, in the base $-3 + i$ with decimal digits, $10 = (1540)_{-3+i}$, so that when the total in adding one column exceeds 10, 154 has to be carried to the next three columns. Sometimes these carry digits can accumulate into an infinite sequence of digits; however, the final answer is still finite.

See also Bases, Negative; Binary (Dyadic) System; Decimal (Hindu-Arabic) System; Fractals; Hexadecimal System

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WILLIAM J. GILBERT

BASES, NEGATIVE

Amusing extension of the usual positional number systems. It is well known that all the positive integers can be represented in any positive base n (if $n > 2$), using the digits $0, 1, \ldots, n-1$; the decimal, binary, and hexadecimal systems being the most common. However, it is possible to represent all the integers in a negative base $-n$ (if $n > 2$), using the digits $0, 1, \ldots, n-1$. Each positive and negative integer can be represented uniquely in each of these bases, without using a minus sign. For example, -532 can be represented as 1548 in the base -10, since $1 \times (-10)^3 + 5 \times (-10)^2 + 4 \times (-10) + 8 = -532$. The usual arithmetic operations can be extended to these bases, and the exploration of these bases would make a good project for a student. Numbers in the base -10 were first considered by Vittorio Grunwald in an obscure journal in 1885; see Knuth (1973, Ch. 4.1) for further details.

The general number $a_m a_{m-1} \ldots a_2 a_1 a_0$ in the base $-n$ represents $a_m \times (-n)^m + a_{m-1} \times (-n)^{m-1} + \cdots + a_2 \times (-n)^2 + a_1 \times (-n) + a_0$. It will represent a positive number if m is even and a negative number if m is odd. Examples in negative binary (base -2) and negative decimal (base -10) follow:

Decimal	**Negative Binary**	**Negative Decimal**
−11	110101	29
−10	1010	10
−2	10	18
−1	11	19

To convert a decimal to a negative base $-n$, repeatedly divide by the negative base, making sure that the remainder is a nonnegative number less than n. To convert -237 to negative decimal, for example, we perform the following divisions:

$$-237 = 24(-10) + 3$$

$$24 = (-2)(-10) + 4$$

$$-2 = 1(-10) + 8$$

$$1 = 0(-10) + 1$$

Decimal	**Negative Binary**	**Negative Decimal**
0	0	0
1	1	1
2	110	2
10	11110	190

Hence, the decimal -237 can be written as 1843 in negative decimal.

Arithmetic in these bases has some unusual features. In negative decimal, for example, 10 is represented by 190. Therefore, in negative decimal whenever 10 has to be carried, 19 has to be carried to the next two columns. Sometimes these carries accumulate into an infinite series of carry digits; this will always happen if a number is added to its negative. The following example shows the addition of the negative decimal numbers 25 and 58 to obtain the negative decimal 63; that is, $(-15) + (-42) = -57$ in decimal. The carry digits are written at the bottom.

				2	5
+				5	8
. . .	0	0	0	6	3
			1	9	
		1	9		
. . .					

In class, students should construct tables of positive and negative numbers in negative binary and negative decimal. They should then explore the arithmetic operations in these bases; see Nelson (1967) and Gilbert and Green (1979). They can then try to write fractional numbers in these bases.

See also Bases, Complex; Binary (Dyadic) System; Decimal (Hindu-Arabic) System; Hexadecimal System

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WILLIAM J. GILBERT

BINARY (DYADIC) SYSTEM

A method for writing numbers in the base 2, using only 0 and 1 as digits. For example, 1101 represents $1 \times 2^3 + 1 \times 2^2 + 0 \times 2 + 1$, which is 13 in decimal notation. If different bases are being used, the base is often added as a subscript, so this binary number would be written as 1101_2. The general binary number $r_m \cdots r_2\, r_1\, r_0$ represents the sum $r_m \times 2^m + \ldots + r_2 \times 2^2 + r_1 \times 2^1 + r_0 \times 2^0$. Every positive integer can be written in the binary system. For example, $1 = 1_2$, $2 = 10_2$, $3 = 11_2$, $4 = 100_2$, $5 = 101_2$, $10 = 1010_2$, and $13 = 1101_2$.

Some ancient systems of measurements used a rudimentary form of the binary system; a remnant of these systems still in use today is the liquid measure 2 pints = 1 quart. In 1703, Gottfried Leibniz, the co-inventor of the calculus, was the first to make systematic use of the binary system. Leibniz said that binary arithmetic was not intended for practical calculations, but only for illustrating patterns in number systems. However, Leibniz did outline a design for a mechanical calculating machine using binary arithmetic, although it was never built. See Asimov (1977, Ch. 2), Knuth (1973, Ch. 4.1), and Resnikoff and Wells (1984, Ch. 1) for more about the history of binary and other positional number systems.

It was only with the advent of digital electronic computers in the 1950s that binary arithmetic became very practical. The binary system is used to represent numbers internally in a computer, as the digits 1 and 0 can correspond to current flowing or not flowing. Circuits can easily be built that will add and multiply numbers in binary. In computer science, the digits 0 and 1 are usually called bits. A byte often refers to eight bits; thus bytes are the binary numbers from 00000000 to 11111111, or from 0 to 255 in decimal. An alphanumeric character in a computer is usually encoded as one byte, so that 1 kilobyte will hold about a thousand characters, and 1 megabyte will hold about a million characters.

A positive integer is converted from decimal notation to binary notation by repeatedly dividing the number by 2, saving the remainders. For example, the following table converts 23 to binary. The binary representation is the sequence of remainders, written in the reverse order, namely 10111_2.

Divide	**Quotient**	**Remainder**
23/2	11	1
11/2	5	1
5/2	2	1
2/2	1	0
1/2	0	1

Arithmetic can be performed in the binary system just as it is done in the decimal system. In fact, it is easier, since the addition and multiplication tables are much simpler (see Figure 1). The only disadvantage is that a number requires more digits when written in binary, as opposed to decimal. For example, consider the addition and multiplication of 110_2 and 11_2, that is, $6 + 3$ and 6×3. (See Figure 2.)

+	0	1
0	0	1
1	1	10

Addition Table

×	0	1
0	0	0
1	0	1

Multiplication Table

Figure 1 *Tables showing addition and multiplication.*

```
     1 1 0
+      1 1
----------
   1 0 0 1
```

```
      1 1 0
×       1 1
-----------
      1 1 0
    1 1 0
-----------
  1 0 0 1 0
```

Figure 2 *Illustration of binary addition and multiplication.*

Just as fractional numbers can be written in the decimal system using a decimal point, fractional numbers can be represented in binary using a binary (or radix) point, such as $.001_2 = 2^{-3} = \frac{1}{8}$. For example, converting from binary to decimal notation, $1.011_2 = 1 + 0 \times 2^{-1} + 1 \times 2^{-2} + 1 \times 2^{-3} = 1 + \frac{1}{4} + \frac{1}{8} = \frac{11}{8} = 1.375$.

A decimal number can be converted into binary by first converting the integer part, as shown above, and then converting the fractional part. The fractional part is converted by repeatedly multiplying by 2 and separating the integer and fractional parts of the result. The binary expansion will be the list of digits in the integer part, in the order generated. For example, the binary expansion of 0.69 to four binary places is found as follows:

Multiply	**Integer Part**	**Fractional Part**
$.69 \times 2$	1	.38
$.38 \times 2$	0	.76
$.76 \times 2$	1	.52
$.52 \times 2$	1	.04

Hence 0.69 is 0.1011_2, correct to four binary places. Its exact value will be an infinite binary expansion.

In class, students could construct tables of decimal and binary numbers, and they should compare, for example, the relationship between the representations of 0 to 15 with those from 16 to 31. They can construct punched cards for the first thirty-one numbers, with each hole corresponding to a digit 1. Students should also try to develop the rules for addition, subtraction, and multiplication in binary. Note that some calculators have a binary mode.

See also Bases, Complex; Bases, Negative

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WILLIAM J. GILBERT

CODING THEORY

A recent branch of mathematics used to design methods for reliable digital data transmission. These methods should detect, and possibly correct, errors that occur during transmission. The errors may occur when data is passed through an electrical circuit, such as a computer, or during transmission by modems on noisy telephone lines, or via satellite. Errors may occur when data is read from magnetic tape, compact discs, or bar codes. Errors also may occur when a human being copies a long identification number, such as a driver's license.

The need for coding theory resulted from the miniaturization of solid state devices that started in the 1960s. Coding theory is now a very active area of mathematical research that uses a variety of techniques from modern algebra including group theory, linear algebra, and polynomials over finite fields. (The term *coding theory* does not usually include cryptography, which is the science of sending secret messages that can only be read by an authorized user.)

There are two basic types of codes used: error-detecting and error-correcting codes. Error-detecting codes are simpler, and alert the machine, or user, to an error that has occurred. The data must then be reread or retransmitted. In some situations however, it is not possible to resend the data, and error-correcting codes are needed. For example, if data is stored on magnetic tape, and is found to be corrupt months later when the tape is read, the original data may no longer exist. Error-correcting codes also are used on compact discs so that the music or data can be read correctly, even if there is some dust or a scratch on the disc.

One example of coding in everyday use is the International Standard Book Number (ISBN). Each published book is given an ISBN consisting of ten digits, arranged in four groups, such as 0–123–45678–9. The first digit is a code for the language; 0 stands for English, 2 for French, etc. The second group of digits is a code for the publisher, and the third group is the publisher's number for the book. The last digit is one of 0, 1, 2, . . . , 9, X, and is a check digit that helps detect any errors when the ISBN is copied when ordering the book. This check digit is chosen so that for any valid ISBN, $a_1\ a_2\ a_3 \ldots a_9\ a_{10}$, the sum $1a_1 + 2a_2 + 3a_3 + \cdots + 9a_9 + 10a_{10}$ is divisible by 11, or equivalently, congruent to 0 modulo 11. This means that the check digit is congruent to $1a_1 + 2a_2 + 3a_3 + \cdots + 9a_9$ modulo 11. The check digit X stands for the number 10. This code will detect any single error or a transposition error. Thus, if one digit is copied incorrectly, or if two digits are transposed, then the result will not be a valid ISBN, and a wrong book would not be ordered. See the two articles by Gallian (1991a, b) for descriptions of other checks used in identification numbers.

One simple way to detect errors in electrical circuits is to add a single parity check digit. Suppose that the basic unit inside a computer is a byte, consisting of eight binary digits. Then one additional digit can be attached to each byte to form nine binary digits called a code word. This extra digit is chosen so that there are an even number of ones in each code word, and so is called a parity check digit. For example, if 10110110 and 00110000 are two bytes, then the code words, including a parity check digit at the right end, would be 101101101 and 001100000. This coding will detect any error in a single digit, because the parity of the number of ones will change if a single 1 is changed to a 0, or if a single 0 is changed to a 1. The computer uses the first eight digits to determine which byte a word represents. There are circuits that check each word to determine if it contains an even number of ones. If the circuit detects an odd number of ones, it triggers an alarm called a parity check error, because there must have been a malfunction somewhere in the computer. This parity check cannot be used to correct the byte in case of an error, and it will not detect two simultaneous errors in one word. By adding further check digits, however, it is possible to correct some errors. See Salwach (1988) and Thompson (1983) for more information on error-correcting codes, and McEliece (1985) to see how these are applied to computer memory chips.

In a class on modular arithmetic, each student could find the ISBN from the back cover of any book, omit the check digit or any other digit, and hand it to another student who would determine the missing digit. Alternatively, two of the digits could be transposed and the other student would have to try to find the correct ISBN; in some cases, this is not always possible. Since most codes are linear, students can use matrix multiplication over the field of two elements to generate examples of error-detecting and error-correcting codes (Thompson 1983).

See also Binary (Dyadic) System

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WILLIAM J. GILBERT

HEXADECIMAL SYSTEM

A method of writing numbers in base 16, instead of the usual decimal system using base 10. The digits used in the hexadecimal system are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, and F, where A stands for 10, B for 11, C for 12, D for 13, E for 14, and F for 15. For example, the hexadecimal number 5D4 denotes the number $5 \times 16^2 + 13 \times 16 + 4$, which is the decimal number 1,492. We often add the base as a subscript, if a number is not written in decimal, so this hexadecimal would be written as $5D4_{16}$. Hexadecimal numbers are also called *radix sixteen numbers.* The general hexadecimal number $r_m \ldots r_2r_1r_0$ represents $r_m \times 16^m + \cdots + r_2 \times 16^2 + r_1 \times 16^1 + r_0 \times 16^0$. (See Knuth 1973, Ch. 4.1 and Resnikoff and Wells 1984, Ch. 1, for the history of positional number systems.)

The hexadecimal system only became popular when digital computers started representing numbers internally in binary format. The advantage of the hexadecimal system is that it can represent these numbers more compactly than binary. It is easy to convert numbers between their hexadecimal and binary forms. Each hexadecimal digit corresponds to four binary digits; $0_{16} = 0000_2$, $1_{16} = 0001_2$, $2_{16} = 0010_2$, $3_{16} = 0011_2$, $4_{16} = 0100_2$, $5_{16} = 0101_2$, $6_{16} = 0110_2$, $7_{16} = 0111_2$, $8_{16} = 1000_2$, $9_{16} = 1001_2$, $A_{16} = 1010_2$, $B_{16} = 1011_2$, $C_{16} = 1100_2$, $D_{16} = 1101_2$, $E_{16} = 1110_2$, and $F_{16} = 1111_2$. Binary numbers can be converted to hexadecimal by grouping the binary digits in fours, starting at the binary point, and then replacing each group of four binary digits by the corresponding hexadecimal digit. For example, $110\ 1111\ 0010\ 1100_2 = 6F2C_{16}$.

It is a little more complicated to convert a whole number from the decimal form to the hexadecimal form. This is achieved by repeatedly dividing by the base 16 until there is no remainder. Consider the decimal number 3,236. Dividing by 16 gives $202 \times 16 + 4$. Then dividing the quotient 202 again by 16 gives $12 \times 16 + 10$. The process is continued in this way until there is no remainder. The remainders in these divisions are the hexadecimal digits.

$$\begin{aligned} 3{,}236 &= 202 \times 16 + 4 \\ &= (12 \times 16 + 10)16 + 4 \\ &= 12 \times 16^2 + 10 \times 16 + 4 \\ &= CA4_{16} \end{aligned}$$

Arithmetic can be performed directly using hexadecimal numbers, but you have to know the addition and multiplication tables for all pairs of numbers up to 15. For example, $7_{16} + D_{16} = 7 + 13 = 20 = 14_{16}$ and $7_{16} \times D_{16} = 7 \times 13 = 91 = 5B_{16}$. Some calculators have options for performing hexadecimal arithmetic.

Just as fractional numbers can be represented in the decimal system by using expansions with a decimal (or radix) point, fractional numbers can be represented in the hexadecimal system using expansions with a radix point. The decimal equivalent of $2E.3F_{16}$ is thus $2 \times 16 + 14 + 3 \times 16^{-1} + 15 \times 16^{-2} = 46 + (63/256) = 46.24609375$.

Students should construct addition and multiplication tables for hexadecimal digits. They can also explore number systems in other bases and discuss the advantages and disadvantages of different bases.

See also Bases, Complex; Bases, Negative; Binary (Dyadic) System

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WILLIAM J. GILBERT

MAGIC SQUARES

Mathematical curiosities that have been known for over 4,000 years. A magic square of order n consists of a sequence of distinct integers arranged in an n by n square array so that the sum of the numbers in every row, in every column, and in each main diagonal is the same. This sum is called the *magic constant* of the square. Usually, the integers used are the numbers from 1 to n^2, and the square is then called a *traditional magic square.* Magic squares exist for all orders larger than 2. For example, the nine integers in the first square in Figure 1 form a traditional magic square of order 3, with magic constant 15. The second square is a traditional magic square of order 4, with magic constant 34. The magic constant in a traditional magic square of order n must be $n(n^2 + 1)/2$. This follows because the sum of all the numbers from 1 to n^2 is $n^2(n^2 + 1)/2$, and each of the n rows must have the same sum. According to an ancient Chinese legend, the Emperor Yu, who lived around 2200 B.C., discovered a mystic turtle on the banks of the Yellow River. The markings on the turtle's back looked like dots that formed a third-order magic square. A treatise in Arabic on magic squares was written in the ninth century A.D. In 1514, Albrecht Dürer produced an engraving entitled "Melancholia" that contained a magic square of order 4, with the numbers 15 and 14 appearing as the two central numbers on the bottom row.

2	9	4
7	5	3
6	1	8

10	16	1	7
3	5	12	14
8	2	15	9
13	11	6	4

Figure 1

One method of constructing magic squares of order n is to use numbers written in the base n. For example, the magic square of order 4 in Figure 1 was constructed from the first square in Figure 2, consisting of the numbers from 00 to 33 in base 4. The right (units) digits of each row, column, and diagonal form an arrangement of the four possible digits 0, 1, 2, 3. The left digits also form an arrangement of the four digits, and so the sum of each row, column, and diagonal must be the same. Furthermore, each number is a different combination of the four digits, and so the numbers in the square are 0 to 15, in some order. The magic square is now obtained by converting each number from base 4 to the decimal system and then adding one to each number.

21	33	00	12
02	10	23	31
13	01	32	20
30	22	11	03

Q♡	J♣	A♠	K◇
A◇	K♠	Q♣	J♡
K♣	A♡	J◇	Q♠
J♠	Q◇	K♡	A♣

Figure 2

This 4 by 4 square consisting of all the pairs of elements from 0 to 3 can also be used to solve the well-know problem of arranging the sixteen court cards from a deck of playing cards in a square so that no two cards in any row, column, or diagonal belong to the same suit or have the same rank. The first digits are replaced by the ace, king, queen, and jack, while the second digits are replaced by the four suits. There are many other methods for constructing magic squares, and the methods for constructing odd- and even-ordered squares are usually quite different. Various methods of construction are given in Ball and Coxeter (1987, Ch. 7), Benson and Jacoby (1976), Eiss (1988), and Kraitchik (1953).

The left digits, written separately, of the first 4 by 4 square in Figure 2 and the right digits, written separately, are examples of an interesting mathematical object called a *Latin square* (see Figure 3). A Latin square of order n is an n by n array of n numbers arranged so that each number appears exactly once in each row and once in each column. The two Latin squares obtained in Figure 3 are called *orthogonal* because, when they are superimposed, each number of the first square occurs exactly once with each number of the second square. Latin squares are used in different mathematical areas such as designing statistical experiments and in coding theory (Stein 1976, 243–263). Any two orthogonal Latin squares of order n, using the numbers 0 to $n - 1$, can be combined in the manner above to produce a magic square, so long as each diagonal of both Latin squares contains every number exactly once.

In 1779, Leonhard Euler, in a paper called "On a new type of magic square," posed the famous problem of the thirty-six officers from six ranks and six regiments. He claimed that it was impossible to arrange these officers on parade in a 6 by 6 square so that each row and each column contained one officer from each rank and one from each regiment. For this reason, orthogonal Latin squares are sometimes called Euler squares. In 1899, it was proved, by exhaustive enumeration, that the problem of the thirty-six officers was insoluble. However, in 1959 it was shown that orthogonal Latin squares exist for all orders larger than 6 (Stein 1976, 243–263).

As an exercise in elementary addition, the class should construct all possible magic squares of order 3 using a given set of numbers. A comparison of the answers will lead to a discussion of the symmetries of a square. As an exercise in conversion from base 4, students should first try to solve the problem of arranging the sixteen court cards, or four different-colored cards containing four pictures. Then the class should construct Latin squares of order 4 and determine which are orthogonal. Finally, these orthogonal Latin squares can be used to solve the court card problem. Students could also be asked to show why there is no magic square of order 2.

See also Enrichment, Overview; Recreations, Overview

2	3	0	1
0	1	2	3
1	0	3	2
3	2	1	0

1	3	0	2
2	0	3	1
3	1	2	0
0	2	1	3

Figure 3

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WILLIAM J. GILBERT