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Arithmetic in Complex Bases

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In this note, we introduce a novel way of doing complex arithmetic that does not involve separating the complex numbers into their real and imaginary parts. This method uses the representation of complex numbers in positional notation using a complex base $-n+i$, for a positive integer n, with natural numbers as digits. Addition, subtraction and multiplication can be performed directly in this positional notation and is similar to real decimal arithmetic; the main difference is in the carry digits. However, division is more complicated and the construction of a good algorithm for long division is a challenging unsolved problem.

We say that an integer z (real or complex) is **represented in the base** b with **digits** from the set $\mathfrak{D}$ if it is written in the form

$$z = a_k b^k + a_{k-1} b^{k-1} + \cdots + a_1 b + a_0,$$

where $a_k, a_{k-1}, \ldots, a_1, a_0 \in \mathfrak{D}$. We denote this representation of z by the **positional notation** $(a_k a_{k-1} \ldots a_1 a_0)_b$. It is well known that the natural numbers can be represented in any integral base $b > 1$ using the digit set $\mathfrak{D} = \{0, 1, 2, \ldots, b-1\}$ and arithmetic can be performed in any of these bases; of course, the decimal (base 10) and binary (base 2) representations are the most popular. All the real integers, both positive and negative, can be uniquely represented by means of a negative integral base $-b < -1$ using the natural number digit set $\mathfrak{D} = \{0, 1, 2, \ldots, b-1\}$. Reference **[4]** contains details of the arithmetic in these negative bases.

The integers in the field of complex numbers, called **Gaussian integers**, are of the form $x + iy$, where x and y are real integers. For each fixed positive integer n, Kátai and Szabó **[5]** proved that *all the Gaussian integers can be uniquely represented in the base* $b = -n + i$ *using the digit set* $\mathfrak{D} = \{0, 1, 2, \ldots, n^2\}$. They also showed that these bases and their conjugates are the only possible ones in which the digit set consists of the natural numbers $0, 1, 2, \ldots, \text{Norm}(b) - 1$.

The base $b = -1 + i$ provides a binary representation of the complex numbers using 0 and 1 as digits; for example

$$(1011)_{-1+i} = (-1+i)^3 + (-1+i) + 1 = 2 + 3i,$$

$$(1100)_{-1+i} = (-1+i)^3 + (-1+i)^2 = 2.$$

The base $b = -3 + i$ yields a decimal representation using the digit set $\mathfrak{D} = \{0, 1, 2, \ldots, 9\}$; for example, $5 + 6i$ is written in positional notation as $(1443)_{-3+i}$. An efficient method for converting a number into a complex base will be given later. See **[6**, § 4.1] for further details of the history of negative and complex bases.

Addition and multiplication of two numbers written in positional notation in base $-n+i$ can be performed in the same way as real arithmetic in base n^2+1, except for a change in the carry digits. The allowable digits in base $-n+i$ are $0, 1, 2, \ldots, n^2$, so whenever the sum of one column exceeds n^2, then n^2+1, or some multiple of it, has to be carried to the higher columns. Since $n^2 + 1 = (1 \;\; 2n-1 \;\; (n-1)^2 \;\; 0)_{-n+i}$, an overflow of n^2+1 in one column means that the digits $1 \;\; 2n-1 \;\; (n-1)^2$ have to be carried to the next *three* higher columns. The following examples illustrate some of this arithmetic. For clarity the subscripts for each base have been omitted in the displayed calculations, and the Cartesian form $z = x + iy$ of the numbers is shown alongside the complex base calculation. The carry digits are placed beneath the sum. There are various ways to subtract in the base $-n+i$; one method is to multiply the subtrahend by negative one and then add.

EXAMPLE 1. The calculations below illustrate addition and multiplication of $2+3i$ and $-1-i$ in base $-1+i$.

$$\begin{array}{r} 1011\\ +\ \ 110\\ \hline 1110101\\ \hline 110\\ 110 \end{array}\qquad \begin{array}{r} 2+3i\\ -1\ \ -i\\ \hline 1+2i\\ \hline \end{array}\qquad \begin{array}{r} 1011\\ \times\ \ 110\\ \hline 10110\\ 101100\\ \hline 11101001010\\ \hline 110\\ 110\\ 110\\ 110 \end{array}\qquad \begin{array}{r} 2+3i\\ \times\ \ -1\ \ -i\\ \hline \\ \\ 1\ -5i\\ \hline \end{array}$$

EXAMPLE 2. The calculations below give an arithmetical check that $i^2+1=0$ in base $-3+i$.

$$\begin{array}{r} 13\\ \times 13\\ \hline 130\\ 39\\ \hline 169\\ \hline \end{array}\qquad \begin{array}{r} i\\ \times\ \ i\\ \hline \\ \\ i^2\\ \hline \end{array}\qquad \begin{array}{r} 169\\ +\qquad 1\\ \hline \ldots 0000000\\ \hline 154\\ 154\\ 154\\ 154\\ 154\\ 154\\ \cdot^{\cdot^{\cdot}} \end{array}\qquad \begin{array}{r} i^2\\ +\ \ 1\\ \hline 0\\ \hline \end{array}$$

The addition shown in Example 2 illustrates a problem that arises in negative and complex bases. This is the fact that there can be an infinite series of carry digits, even though the sum is finite. This phenomenon must always happen whenever a number and its negative can be represented in the same base using natural numbers as digits. (See also [4].) This infinite sequence of carry digits does not invalidate the arithmetic because the carry numbers all sum to zero after a certain stage. In the above example, the numbers inside the dotted triangle sum to zero.

It can be proved that the number of digits in the sum of two numbers expressed in the base $-n+i$ is at most three more than the number in the largest summand if $n \geqslant 4$; at most five more if $n=2$ or 3 and at most eight more if $n=1$. The following examples show the extreme cases in the bases $-1+i$ and $-3+i$.

EXAMPLE 3. The calculations below show additions with long totals in bases $-1+i$ and $-3+i$ respectively.

$$\begin{array}{r} 1011\\ +\quad 1011\\ \hline 111010010100\\ \hline 110\\ 110\\ 110\\ 110\\ 110\\ 110\\ 110 \end{array}\qquad \begin{array}{r} 2+3i\\ 2+3i\\ \hline 4+6i\\ \hline \end{array}\qquad \begin{array}{r} 905\\ +\quad 655\\ \hline 15609090\\ \hline 154\\ 154\\ 154\\ 154 \end{array}\qquad \begin{array}{r} 77-54i\\ 38-31i\\ \hline 115-85i\\ \hline \end{array}$$

This addition could be automated to add two k-digit numbers in base $-3+i$ and to stop after $k+5$ digits. The sum obtained would be correct regardless of whether any carry digits remain; the extra digits must sum to zero.

The propagation of carry digits in real arithmetic has been a concern of computer scientists who are trying to speed up arithmetical operations. This problem is accentuated in complex base arithmetic. One suggested solution for real arithmetic [1] extends very neatly to complex bases and avoids infinite carries. Each representation of an integer in the base b can be viewed as a polynomial in b. The arithmetical operations of addition, subtraction and multiplication can be first done, without carries, in the ring of formal polynomials $\mathbf{Z}[b]$ by permitting the coefficients of the powers of b to be any integer, not just the allowable digits for the base. At the end of all the calculations the resulting polynomial can be "cleared," using the minimum polynomial of b, so that all the coefficients are digits lying in the correct range. In the examples which follow, a polynomial $a_k b^k + \cdots + a_1 b + a_0 \in \mathbf{Z}[b]$ will be written in positional notation as $(a_k \dots a_1 a_0)$. The minimum polynomial of $-n+i$ is $b^2+2nb+n^2+1$ or $(1\ 2n\ n^2+1)$ in base b positional notation; adding any multiple of this minimum polynomial to any other polynomial will not affect its value when $b=-n+i$ because $b^2+2nb+n^2+1=0$. The **clearing algorithm** is as follows. Let a_r be the coefficient of the smallest power of b which lies outside the range from 0 to n^2. Then there exists an integer s such that $0 \leqslant a_r + s(n^2+1) \leqslant n^2$. Add sb^r times the minimum polynomial to clear this rth coefficient. The following example shows this algorithm in operation using the base $-3+i$ whose minimum polynomial is (1 6 10). Because the *minimum* polynomial of the base is used in this process, the algorithm will terminate after a finite number of steps [3] and hence this method avoids any infinite series of carries.

EXAMPLE 4. The following calculations show the multiplication $(182)_{-3+i} \times (38)_{-3+i} = (13546)_{-3+i}$ using the clearing algorithm.

$$\begin{array}{rrrrr}
 & & 1 & 8 & 2 \\
 & \times & & 3 & 8 \\
\hline
 & & 8 & 64 & 16 \\
 & 3 & 24 & 6 & \\
\hline
 & 3 & 32 & 70 & 16 \\
\hline
 & & -1 & -6 & -10 \\
 & -6 & -36 & -60 & \\
1 & 6 & 10 & & \\
\hline
1 & 3 & 5 & 4 & 6 \\
\hline
\end{array}
\qquad
\begin{array}{l}
\text{polynomial} \\ \text{multiplication} \\ \\ \text{clearing} \\ \text{algorithm}
\end{array}
\qquad
\begin{array}{r}
-14+2i \\
\times -1+3i \\
\hline
\\ \\ \\ \\ \\
=8-44i \\
\hline
\end{array}$$

The easiest way to convert any Gaussian integer into a complex base is to use this clearing algorithm. A Gaussian integer $s+it$ can be written as $t(-n+i)+(s+nt)$ or as $(t\ s+nt)$ in positional notation in base $-n+i$. The clearing algorithm will now convert this to a representation in base $-n+i$ with the digits in the proper range from 0 to n^2.

EXAMPLE 5. The following calculations illustrate the conversion of $5+6i$ into its base $-3+i$ representation, $(1443)_{-3+i}$.

$$\begin{array}{rrrrl}
 & & 6 & 23 & =5+6i \\
 & -2 & -12 & -20 & \\
1 & 6 & 10 & & \\
\hline
1 & 4 & 4 & 3 & \\
\hline
\end{array}$$

The positional representation of the Gaussian integers in base $b=-n+i$ can be extended to cover all the complex numbers by using infinite radix expansions. Each complex number can be written as a convergent sum

$$\sum_{j=-\infty}^{k} a_j b^j, \tag{1}$$

where the digits $a_j \in \{0, 1, 2, \ldots, n^2\}$. The expansion (1) is written in positional notation, using a radix point, as $(a_k a_{k-1} \ldots a_0 . a_{-1} a_{-2} \ldots)_b$. For example,

$$(5+i)/4 = 1 + (-1+i)^{-2} + (-1+i)^{-3} = (1.011)_{-1+i},$$
$$\sqrt{2} + i = (15.49778016\ldots)_{-3+i},$$

and

$$1/3 = (1.\overline{4724})_{-3+i},$$

where the bar over a string of digits indicates that they are to be repeated indefinitely.

As in real systems, complex numbers of the form $x + iy$, with x and y rational, have periodic or terminating expansions in base $-n+i$. All the other complex numbers have aperiodic expansions. Also, as in real systems, these expansions are not always unique. In complex bases some numbers have one expansion, some two, and a few even have three different expansions. For example, $(1-2i)/5 = (0.\overline{001})_{-1+i} = (1.\overline{100})_{-1+i} = (111.\overline{010})_{-1+i}$. Reference [2] discusses the geometric significance of the points with multiple expansions.

Long division in real arithmetic consists of dividing one finite expansion by another. By shifting the radix point of the divisor and dividend this is equivalent to dividing one integer by another. The long division algorithm in real arithmetic for dividing one natural number c by d in the positive base b is as follows. Initially set $c = ad + r_0$, where $0 \leqslant r_0 < d$, and a is an integer. Then, for $j > 0$, let

$$br_{-j+1} = a_{-j} d + r_{-j}, \text{ where } 0 \leqslant r_{-j} < d.$$

This defines a sequence of digits a_{-j}, which automatically lie in the required range from 0 to $b-1$, and then $c/d = (a . a_{-1} a_{-2} \ldots)_b$.

This long division algorithm can be extended to complex bases to divide one Gaussian integer by another. However, the allowable remainders, r_{-j}, can be complex and they form a complicated set that depends on both the divisor and the base. That is, for each Gaussian integer divisor d and for each complex base b, there is some **remainder set** $\mathfrak{R}(d; b)$ of Gaussian integers such that the long division algorithm will yield a convergent radix expansion in base $b = -n + i$ with $0 \leqslant a_{-j} \leqslant n^2$ if and only if $r_{-j} \in \mathfrak{R}(d; b)$ for all $j \geqslant 0$.

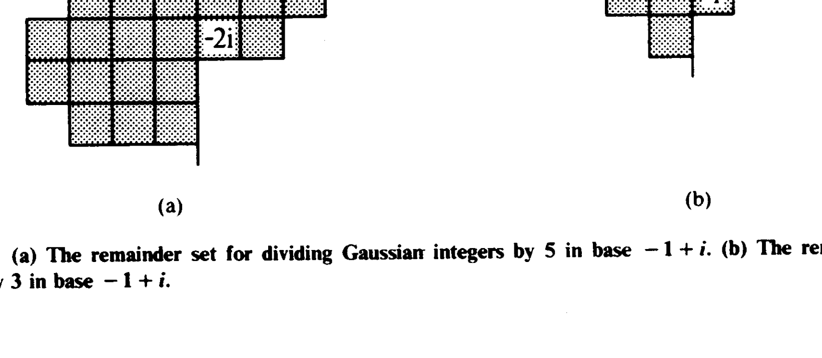

FIGURE 1. **(a) The remainder set for dividing Gaussian integers by 5 in base $-1+i$. (b) The remainder set for dividing by 3 in base $-1+i$.**

FIGURE 1 shows two examples of these remainder sets, namely $\mathfrak{R}(5; -1+i)$ and $\mathfrak{R}(3; -1+i)$. In the case of division by 3 in base $-1+i$ the remainder set $\mathfrak{R}(3; -1+i)$ forms a complete residue system modulo 3; that is, this set tiles the plane by translations along Gaussian integers multiplied by 3. Therefore, at each stage of the long division algorithm, the remainder r_{-j} is uniquely determined and the resulting radix expansion will be unique.

In the case of division by 5 in base $-1+i$, the remainder set has 38 elements, which is more than the norm of the divisor. This means that in division by 5, there will sometimes be a choice for the remainder and so the algorithm will sometimes yield more than one radix expansion of the same number. This happens in the calculation of $(4+2i) \div 5$ in base $-1+i$. There is a choice of two remainders at the initial stage; either $4+2i=(1)5+(-1+2i)$ or $4+2i=(1+i)5+(-1-3i)$. After that, the algorithm is uniquely determined. The first alternative yields the following expansion.

$$\begin{aligned} 4+2i &= 1.5+(-1+2i) \\ (-1+2i)(-1+i) = -1-3i &= 0.5+(-1-3i) \\ (-1-3i)(-1+i) = 4+2i &= 1.5+(-1+2i) \end{aligned}$$

The algorithm now repeats and so $(4+2i)/5=(1.\overline{01})_{-1+i}$. The second alternative yields a different expansion.

$$\begin{aligned} 4+2i &= (1+i)5+(-1-3i) \\ (-1-3i)(-1+i) = 4+2i &= 1.5+(-1+2i) \\ (-1+2i)(-1+i) = -1-3i &= 0.5+(-1-3i) \end{aligned}$$

The algorithm now repeats and, since $1+i=(1110)_{-1+i}$, it follows that $(4+2i)/5=(1110.\overline{10})_{-1+i}$. The reader should try calculating the three expansions of $(-3-4i)/5$ in base $-1+i$. Each periodic expansion of period p can be evaluated by the standard method of multiplying by the pth power of the base and then subtracting the original expansion from it.

The above long division algorithm depends on first calculating the remainder sets $\mathfrak{R}(d; b)$. Even though bounds can be put on their size, the exact determination of these remainder sets appears to be a tedious task. Are there other ways of doing division? The method of division given in [**1**] does not extend to complex bases. It essentially consists of finding inverses in the formal power series ring $\mathbf{Z}[[b^{-1}]]$ and then using the clearing algorithm. However, the inverse power series do not always converge when a complex base is substituted for b.

The complexity of the division can be appreciated by looking at the set of points which have the same initial expansion in a given complex base [**2**]. These subsets of the complex plane have fractal boundaries and it is not an easy task to determine whether a ratio of two Gaussian integers lies in a given set. Can the reader find either an easy method for calculating the remainder sets $\mathfrak{R}(d; b)$ or find an alternative technique for doing division?

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