What we need to learn from Sec 3.3   Given a set S (nothing else is known about it).

(1) Subsets $S_1, \cdots, S_m$ form a partition of S if

(i) S_i are disjoint.

(ii) Union of S_i is S.

In other words, each $a \in S$ belongs to one and only one S_i.

(2) Given any statement on 2 elements a, b (ordered) in S, it induces a relation R on S as:

If statement true, then aRb (say a-is-related-to-b). If statement false, then $a \not R b$ (say a-is-not-related-to-b).

(3) R is call (i) reflexive, if $\forall a \in S$, aRa

(ii) symmetric, if $\forall a, b \in S$, $aRb \Rightarrow bRa$

(iii) transitive, if $\forall a, b, c \in S$, aRb and $bRc \Rightarrow aRc$.

A relation that satisfies (i)-(iii) is called an **equivalence relation.**

(4) Examples of general relations. Take $S = \mathbb{Z}$.

e.g.1 Define aRb iff $a \leq b$.

Then, R reflexive ($a \leq a$), transitive $a \leq b$ and $b \leq c \Rightarrow a \leq c$, but not symmetric (say, $3R4$ but $4 \not R 3$).

e.g.2 Define aRb iff $a < b$.

Same as above, but R no longer reflexive.

e.g.3 Define aRb iff $a|b$.

R reflexive ($a|a$), transitive $a|b$ and $b|c \Rightarrow a|c$ (Prop 3.11 (i)), but not symmetric (say, $2R4$ but $4 \not R 2$). Similar to e.g.1.

e.g.4 Define aRb iff $a + b = 0$.

Then, not reflexive nor transitive but symmetric.

e.g.5 Define aRb iff $a = 2b$.

Then, neither reflexive nor symmetric nor transitive.

(5) Any partition of S induces an equivalence relation via the statement aRb iff a, b in same subset. (Proved in Lec 10.)

(6) Conversely, any equivalence relation induces a partition $S_1, S_2, \cdots$ to S, as follows:

Take 1 element $a_1 \in S$. Collect all b_1's that are related to a_1. Call the collection S_1.

Look at $S - S_1$. Take 1 element $a_2 \in S - S_1$. Collect all b_2's in $S - S_1$ that are related to a_2. Call the collection S_2.

Look at $S - S_1 - S_2$ etc, until all elements in S are placed in some S_i.

Proof that S_i's form a partition is in the textbook.

(7) So, partitions and equivalence relations on S come hand-in-hand.

The S_i's are called **equivalence classes**, and any $b_i \in S_i$ is called **a representative** of S_i.

(8) Intuition, and reasons for terminology.

There are many thing in S, but some are equivalent to another, some are not. We don't quite care which specific one among those equivalent ones. Effectively we only care about which subsets.

e.g. There is a bag of many socks and gloves and hats and scarfs (this is S). I put all socks in drawer 1 (S_1), all lefthanded gloves in drawer 2 (S_2), all righthanded gloves in drawer 3 (S_3), all hats in drawer 4 (S_4), all scarfs in drawer 5 (S_5). Every morning, I pick 2 socks from S_1, 1 glove from each of $S_{2,3}$, one hat from S_4 and one scarf from S_5 before going to school. Don't care which specific item(s) from each drawer, all socks are "equivalent" to one another etc. Each hat is representing my collection of hat.