Consider discrimination of bipartite states:

Richard draws \( x \) up \( \rho_x \), prepares \( \rho_x \in D(X_A \otimes X_B) \)

gives sys \( X_A \) to Alice, sys \( X_B \) to Bob.

Alice and Bob can apply a measurement (in LOCC, or SEP, or PPT, or jointly).

When locality restriction reduces the prob of success,
the ensemble to be discriminated is said to exhibit nonlocality.

Eq1. Let \( X = 0, 1, 2, 3, \ \rho_x = \frac{1}{4} \) for all \( x \).

\( \rho_{0, 1, 2, 3} \) are the 4 Bell states.

- Using a joint measurement, the 4 Bell states can be perfectly discriminated.
- Using a meas in LOCC, SEP or PPT, the 4 Bell states cannot
be perfectly distinguished. To see this, suppose such a Sef meas
exists: This perfect measurement \( \{ M_k \} \) necessarily has
\( M_k \propto \rho_k \)
With this measurement, instead of using it for state discrimina-
Alice & Bob instead:

1. Each prepares local \( \frac{1}{\sqrt{2}}(100+111) \), measures \( \rho_k \)
   obtained by \( cc \).
2. Obtain \( \{ M_k \} = k \)
3. Post-meas state on \( A'B' \)
   \( x \propto M_k \)
   Which is \( \rho_k \).

Then the Sef meas turns a state in \( \text{Ent}_1(A' | A=BB') \) to \( \text{Ent}_2(A' | B') \)
a contradiction. Similar proof holds for PPT meas.
More detailed proof in Sec 19.2.1:

Let $|M_k\rangle = (U_k \otimes 1) \beta_k \in \mathcal{X}_A \otimes \mathcal{X}_B$, where $U_k$ is unitary (so $|M_k\rangle$ maximizes $\langle X_A | X_B \rangle$)

Let $M_k = \sum_j P_{kj} \otimes Q_{kj} \in \text{SEP} (\mathcal{X}_A : \mathcal{X}_B)$ be mean of corr. to $|M_k\rangle$.

Prob succ $= \frac{1}{t} \sum_{k=1}^{t} \frac{1}{n} \cdot \text{tr} (\frac{1}{3} \sum_j P_{kj} \otimes Q_{kj} \cdot \frac{1}{n} \cdot (U_k \otimes 1_B) \beta_k \beta_k^* (U_k^* \otimes 1_B))$

transpose $= \frac{1}{t} \sum_{k=1}^{t} \frac{1}{n} \cdot \text{tr} (\frac{1}{3} \sum_j P_{kj} \otimes Q_{kj} \cdot \frac{1}{n} \cdot (U_k Q_k^t \otimes 1_B) \beta_k \beta_k^* (U_k^* \otimes 1_B))$

partial trace $= \frac{1}{t} \sum_{k=1}^{t} \frac{1}{n} \cdot \text{tr} (\frac{1}{3} \sum_j P_{kj} \cdot \frac{1}{n} \cdot U_k Q_k^t \cdot (\text{trace}_{\mathcal{X}_B} \beta_k \beta_k^*) - U_k)$

$\leq \frac{1}{t} \sum_{k=1}^{t} \frac{1}{n} \cdot \frac{1}{3} \text{tr} (P_{kj} \cdot \text{trace}_{\mathcal{X}_B} \cdot U_k^t Q_k U_k)$

$= \frac{1}{t} \sum_{k=1}^{t} \frac{1}{n} \cdot \frac{1}{3} \text{tr} (P_{kj} \otimes Q_{kj})$

$= \frac{1}{t} \sum_{k=1}^{t} \frac{1}{n} \cdot \frac{1}{3} \text{tr} M_{kj} = \frac{n}{t} \cdot \frac{1}{n^2}$

If $t \geq n+1$, cannot perfectly distinguish with SEP means.

Also if $t = n^2$, prob succ $\leq \frac{1}{n}$ (achievable).
Any 2 orthogonal bipartite pure states can be discriminated perfectly by LUCC.

\[ |1\rangle = |1\rangle_A |1\rangle_B + |2\rangle_A |2\rangle_B + \ldots + |n\rangle_A |n\rangle_B \]

\[ |2\rangle = |1\rangle_A |2\rangle_B + |2\rangle_A |1\rangle_B + \ldots + |n\rangle_A |n\rangle_B \]

Note \( |1\rangle, |2\rangle \) not normalized; can be \( 0 \), not Schmidt decomposes but \( \{ |1\rangle, |2\rangle, \ldots, |m\rangle \} \) orthonormal basis for \( A \).

Let \( \{ \ldots, |m\rangle \} \) be orthonormal basis for \( B \), \( m \leq n \)

\[ |1\rangle_B = \sum_{ij} m_{ij} |i\rangle |j\rangle_B \]

\[ |1\rangle_B = \sum_{ij} g_{ij} |i\rangle |j\rangle_B \]

\( F = \sum \langle i | j | F_{ij} \), \( G = \sum \langle i | j | G_{ij} \)

\[ F G^* = \sum \langle i | j | F_{ij} \langle i' | j' | G_{i'j'} \]

\[ = \sum \langle i | j | j' | i' | F_{ij} \langle i' | j' | \]

\[ = \sum \langle i | j | \sum_{ij'} \langle i | j' | G_{ij} \]

\[ = \sum \langle i | j | n | \langle n | m | = \begin{bmatrix} \langle n | m | - \langle n | m | \langle n | m | \\
\end{bmatrix} \]
Note that $\langle \psi | \psi \rangle = 0 = \sum_{i=1}^{n} \langle \psi_i | \psi_i \rangle = \text{Tr}(F \hat{G}^*)$

Consider a unitary $U$ on $A$ relating the basis $\{|1\rangle, \ldots, |n\rangle\}$ to $\{|1\rangle, \ldots, |n\rangle\}$ as:

$$|i\rangle_A = \sum_{j} \tilde{U}_{ij} |j\rangle_A$$

and define $|i\rangle_B$, $|j\rangle_B$, $\ldots$, $|n\rangle_B$ as

$$|k\rangle_B = \sum_{k=1}^{n} U_{kL} |k\rangle_B$$

Then, $|\psi\rangle = \sum_{i=1}^{n} |i\rangle_A |i\rangle_B = \sum_{i=1}^{n} |i\rangle_A \sum_{k=1}^{n} F_{ik} |k\rangle_B$

$$= \sum_{i=1}^{n} \left( \sum_{j=1}^{n} \tilde{U}_{ij} |j\rangle_A \right) \sum_{k=1}^{n} F_{ik} \sum_{l=1}^{n} U_{kl} |l\rangle_B$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{n} \left( \sum_{i=1}^{n} \sum_{k=1}^{n} \tilde{U}_{ij} F_{ik} U_{kl} \right) |j\rangle_A |l\rangle_B$$

$$= \left( \tilde{U}^* F U \right)_{jl}$$

$$|\psi\rangle = \sum_{j=1}^{n} \sum_{k=1}^{n} \left( \tilde{U}^* F U \right)_{jl} |j\rangle_A |l\rangle_B$$

$$F_{\hat{G}^*} \rightarrow (\tilde{U}^* F U)(\tilde{U}^* F U)^* = U^* \hat{C} \hat{G}^* U$$

and vice-versa, that conjugation of $F_{\hat{g}^*}$ can to local change of basis by Alice & Bob.
Claim: \( \exists U \) st. \( U^* F G^* U \) has equal diagonal entries.

Pf (elementary, see 000 07 098)

But \( \text{tr} \ U^* F G^* U = \text{tr} \ F G^* = 0 \) i.e. all diagonal entries of \( U^* F G^* U \) are zero!

Protocol: Alice measures along \( \{ 1' \}, \{ 2' \}, \ldots, \{ M' \} \) basis,

sends outcome "3" to Bob.

If state was \( |4\rangle \), Bob's state is now

\[
\sum_{\mathcal{E}} (U^* F W)_{\mathcal{E}} |4\rangle_{B}
\]

Orthogonal \( \mathcal{E} \) can be perfectly discriminated by a measurement on \( B \)!!

BONUS: Works for any \( \# \) parties, by letting \( B = \{ \text{party 2, party 3, \ldots} \} \)

Since now the problem for parties in \( B \) is again to discriminate

2 orthogonal pure states.

It takes 1 classical message from party 1 to party 2

\[
1 - - - - - - - - - - - 2 - - - 3
\]

and last party finds out which of \( |0\rangle, |4\rangle \) is given and announced to all other parties.

\# parties + 1 rounds, \( 2 \times \# \) parties messages needed.
Consider the 9 states in $\mathbb{C}^3 \otimes \mathbb{C}^3$:

\[ \begin{pmatrix} 10 \rangle_A & 11 \rangle_A & 12 \rangle_A \\ 10 \rangle_B & + & + \\ 11 \rangle_B & + & - \\ 12 \rangle_B & + & - \end{pmatrix} \]

Call non-locality without entanglement 9864053

1. They form a basis for $\mathbb{C}^3 \otimes \mathbb{C}^3$.
2. Each is a product state.

Perfect measurement for discrimination $\not\leq$ SEP ($\mathbb{C}^3_A : \mathbb{C}^3_B$).

Prob (failure) given LOCC meas $> 10^{-6}$ !

$\uparrow$

a constant

\[ \begin{pmatrix} 10 \rangle_A & (10 \rangle \pm 11 \rangle) \rangle_B \\ 11 \rangle_A & (11 \rangle \pm 12 \rangle) \rangle_B \\ \frac{10 \rangle \pm 12 \rangle}{\sqrt{2}} \rangle_A & \frac{11 \rangle \pm 12 \rangle}{\sqrt{2}} \rangle_B \end{pmatrix} \]

Not only perfect wins $\not\leq$ LOCC

Cannot run converse to perfect meas by LOCC!

\[ \frac{\text{LOCC}}{\text{(clique)}} \not\leq \text{SEP}! \]

NB. Doesn't help to have unlimited # messages,
and unlimited length of those messages.
Back to pure state transformation, but 3 parties.

\( \text{Initial state: } \frac{1}{\sqrt{2}} (1000> + 1111>)_{ABC} \)

Alice & Bob share \( \frac{1}{2} (1000\times 001 + 111\times 111)_{AB} \) sep, no entanglement.

Cannot distill a key that Charlie doesn't know!

But Charlie is not evil....

He is willing to assist Alice & Bob in distillation.

He measures C along the \{1+, 1-\} basis.

\[
\frac{1}{\sqrt{2}} (100> + 111>) = \frac{1}{\sqrt{2}} \left[ \left( \frac{100> + 111>}{\sqrt{2}} \right) \left( \frac{10> + 11>}{\sqrt{2}} \right) + \left( \frac{100> - 111>}{\sqrt{2}} \right) \left( \frac{10> - 11>}{\sqrt{2}} \right) \right]
\]

(1)

(2)

(1) If Charlie gets "1+", Alice & Bob share \( \frac{100> + 111>}{\sqrt{2}} \)

(2) "1-"

So if Charlie tells Bob, he applies \( \frac{1}{\sqrt{2}} \) if Charlie says +/−.

1. Alice & Bob always share \( \frac{100> + 111>}{\sqrt{2}} \)!
If initial state is \( |\psi\rangle = \frac{1}{\sqrt{3}} \left( |100\rangle + |110\rangle + |101\rangle \right)_{ABC} \)

Then Charlie cannot help Alice & Bob distill one ebit.

But... if they only want 2 out of 3 parties to share an ebit, they canapprox the task arbitrarily well!!

In 1106.1208, Chitambar, Lui, & Lo proved that

\[
|\psi\rangle \rightarrow \frac{1}{\sqrt{2}} \left( |100\rangle + |111\rangle \right)_{AB} \quad \text{is in \( \text{LOCC} \) but not in \( \text{"LOCC"} \)}
\]

or

\[
\begin{array}{c}
|\rangle \\
\text{or} \\
\end{array}
\]

\[
\begin{array}{c}
|\rangle \\
\text{or} \\
\end{array}
\]

includes all LOCC with

\[
\text{includes all LOCC with finite # messages and a restricted class of LOCC ops}
\]

with infinitely many messages.

Aha's \( \text{LOCC} \) is NOT a closed set!!
Semi-formal definition of LOCC: Consider discrete quantum instruments, each defined as a family of completely positive maps $\mathcal{E} = \{ \mathcal{E}_j : j \in \Theta \}$, for an index set $\Theta$ that may be finite or countably infinite, and $\sum_j \mathcal{E}_j$ is TP.

- Discrete quantum instruments form a convex set.

- $\mathcal{E}(\rho) = \sum_{j \in \Theta} \mathcal{E}_j(\rho) \otimes I_j \chi_{j1}$

- Distance between $\mathcal{E}$ and $\mathcal{F}$ with common input space $\&$ index sets:

$$\| \mathcal{E} - \mathcal{F} \|_\diamond = \max_{0 \leq \rho \leq 1, j \in \Theta} \| (I \otimes I - I \otimes F_j(\rho)) \|_1$$

- Sequence of instruments $\mathcal{F}_1, \mathcal{F}_2, \ldots$ converges to $\mathcal{E}$ if

$$\lim_{n \to \infty} \| \mathcal{E} - \mathcal{F}_n \|_\diamond \to 0$$

- When input is $m$-partite, instrument $\mathcal{F} = (\mathcal{F}_j : j \in \Theta)$ is 1-local with respect to the party $k$, if $\forall j$, $\mathcal{F}_j = \bigotimes_{a \neq \{1, 2, \ldots, k-1, k+1, \ldots, m\}} \mathbf{E}_{a, j} \otimes \mathbf{E}_{k, j}$

Some channel with
- Input $X_a$, output $Y_a$
- Held by the $a$-th party

Be operationally, party $k$ applies instrument $\{ \mathcal{E}_j : j \in \Theta \}$ and broadcasts $j$ to all other parties.
• $F'$ is **LOCCLinked** to $F = \{ F_j : j \in \Theta \}$

  if $\exists \ G_1, G_2, \ldots, G_m$

  and $\forall k, G_k$ and each vote to party $k$, $G_k = \{ G_{kj} : j \in \Theta_k \}$

  $\exists$ function $N : \Theta \rightarrow \{ 1, 2, \ldots, m \}$ (who is next)

  s.t. $F' = (G_{N(j)} \circ F_j : \Theta \times \Theta_{N(j)})$

  i.e. $F'$ is obtained from adding one round of LO's and CC from party $n(k)$

  for each outcome $k$ from $F$.

  \[ \text{Def of LOCCLinked}: \]

  1. $LO = LOCCLinked$

  2. $F \in LOCCLinked$ if $F$ is way local vote to some party $k$

  3. $F' \in LOCCLinked (n \geq 2)$ if $F'$ is LOCCLinked to some $F \in LOCCLinked$

  4. $F \in LOCCLinked_{\infty}$ if $F \in LOCCLinked$ for some $r \in N = \{ 1, 2, \ldots \}$

  5. $F \in LOCCLinked$ if $\exists (F_1, F_2, \ldots)$ s.t. each $F_r \in LOCCLinked_{\infty}$,

  \[ \text{Fr LOCCLinked to Fr}_{r-1} \quad \forall r \geq 2 \]

  \[ \lim_{r \to \infty} F_r = F \]

  6. $F \in LOCCLinked_{\infty}$ if $\exists (F_1, F_2, \ldots)$ s.t. $F_r \in LOCCLinked_{\infty}$, $\lim_{r \to \infty} F_r = F$. 22
Operationally, LOCC\textsuperscript{r} can be implemented by \( r\) rounds of classical communication (without limit to the size of the messages), LOCC\textsuperscript{IN} = implemented by some finite-round LOCC, but can't say how many rounds.

LOCC: all protocols either finite-round, or approx arbitrarily well by adding more and more rounds (until reaching a desirable accuracy).

LOCC\textsuperscript{IN}: topological closure, improving the accuracy can require completely different \( 1^{st}, 2^{nd}, \ldots \) rounds.

NB: We allow "coarse-graining" of indices as the LOCC instrument progresses through the rounds.

eg. can require infinite intermediate measurement outcomes but finally decide on one of the 9 outcomes for discrimination of \( \mathcal{M} \).

\[
\begin{array}{cccccccc}
\text{LOCC} & \text{LOCC} & \text{LOCC} & \text{LOCC} & \text{LOCC} & \text{LOCC} & \text{LOCC} & \text{SEP} \\
\neq & \neq & \neq & \neq & \neq & \neq & \neq & \\
\end{array}
\]

- Chitambar
- Fortean-Lo
- Chitambar-Wui-Lo
- random distribution
- preparing the ensemble (no entanglement needed)
- is irreversible
- distinguishing is not in LOCC
- \( \mathbb{C} \) cannot distinguish \( \mathbb{W}/\mathbb{O} \) entanglement, despite in SEP.
Some good news:

- LOCC is convex

- If \( F = (E_1, \ldots, E_n) \) (finite final \# of indices)
  
  \[ m \text{-partite, in } \text{LOCC}_m \text{, finite total dimensional input/outputs } (E_e) \]

  then a finite \# of intermediate outcomes suffices

  so finite CC suffices.

- Subset of LOCC with \( m \) final coarse-grained outcomes is compact.

- Much better bound on \# outcomes for even LOCC

  if the goal is NOT to approx an LOCC instrument
  
  but to optimize an LOCC task (e.g. state discrimination)