Q1  (Q super-mp / super-super operator)

So $A \in L(X,Y)$, $\bar{A}$ takes $\bar{y} \in Y$ to $\bar{A}y \in Y$

$\bar{A} \in T(X,Y)$, $\bar{A}$ takes $A \in L(X)$ to $\bar{A}(A) \in L(Y)$

$L(L(X), L(Y))$, $\forall \bar{A}$

$\bar{A} \in C(X,Y)$, $\forall \bar{A}$, $\bar{A}$ takes $\bar{S} \in D(X \otimes \bar{Z})$ to $\bar{A} \bar{S}(\bar{y}) \in D(Y \otimes \bar{Z})$

i.e. $\bar{A}$ is TP & CP.

Here we want to study $S \in L(T(X,Y), T(X',Y'))$.

and what properties ensure $S \in SM(C(X,Y), C(X',Y'))$.

The end goal is to show $S \in SM(C(X,Y), C(X',Y'))$

$\Rightarrow \forall \bar{A} \in C(X,Y)$, $S(\bar{A}) = \bar{x'}$

for some $\bar{E}, \bar{D}, \bar{F}, \bar{G}$ independent of $\bar{A}$.

Two interpretations:

1. $S$ is a forward-assisted code, turning any input channel $\bar{A}$ to a new channel $S(\bar{A})$ with encoder $\bar{E}$, forward communication $\bar{F}$, and decoder $\bar{D}$. 

[Diagram]

- $E$ to $\bar{A}$
- $\bar{A}$ to $D$
- $D$ to $G$
- $E$ to $F$
- $F$ to $G$
(2) $S$ is a quumb:

We can plug any $\chi (X,Y)$ in the slot and the composition is $S(\chi)$.

The assignment is taken from EPL, 83 (2008) 30004 by Chiribella, D’Ariano, Perinotti.

The question is broken into small steps, so you can try without consulting the paper. You can use results from previous steps in later parts.

You can use the paper, since the proofs aren’t exactly easy.

Watch out for the typos and overlap notations: text are not consistent with the assignment writing.

But you must write your own solutions, and in the notations consistent with the assignment writing developed in the course and justify your work based on course materials.

Goal: to digest what we learnt about channels, partial trace, and their representations.
{(a) Super-complete-positivity.}

Let $S \in \text{SM}(C(X,Y), C(X',Y'))$ be arbitrary.
Let $I \in \text{SM}(C(K,H), C(K,H))$ be the identity super-map.

\[ I(\varphi) \equiv \varphi. \]

Suppose $S \otimes I \in \text{SM}(C(X \otimes K, Y \otimes H), C(X' \otimes K, Y' \otimes H))$

\{(X)\}

ie if $\varphi$ is an arbitrary Choi channel from $X \otimes K$ to $Y \otimes H$
then $S \otimes I(\varphi)$ is a Choi channel from $X' \otimes K$ to $Y' \otimes H$.

Let $\mathcal{A} \in \mathcal{L}(\mathcal{L}(Y \otimes X), \mathcal{L}(Y' \otimes X'))$ be the map induced by $S$
on Choi matrices.

\[ \forall \varphi \in \mathcal{A}, \quad \mathcal{A}(\varphi(\varphi)) = \varphi(S(\varphi)). \]

Explain why:

\\[ (X) \implies \forall \mathcal{E}, \quad \mathcal{A}(\mathcal{E}) = \sum_i S_i \mathcal{E} S_i^* \quad \text{for some } S_i \in \mathcal{L}(Y \otimes X, Y' \otimes X'). \]

\[ \text{in } \mathcal{L}(Y \otimes X) \]

3 marks
Suppose $C \in \text{Pos}(Y \otimes X)$ satisfies the following condition.

$$\forall \omega \in C(X,Y), \quad \text{Tr}(C \cdot J(\omega)) = 1$$

Show that $C = I_Y \otimes \rho$ for some $\rho \in D(X)$.

Hint:

1. It suffices to show $\forall E \in \text{Pos}(Y \otimes X), \quad \text{Tr}(C E) = \text{tr}(I_Y \otimes \rho E)$.

2. Pick $E$ s.t. $\text{Tr}E \neq 1_X$, $\text{Tr}E \neq 1_Y$. Then, $E$ is the Choi matrix of a CP map that is not TP.

Now, you want to add $D = \text{Choi matrix of another CP map}$

s.t. $D + E$ is the Choi matrix of a channel (TP)

3. Try $D = 0 \otimes (1 - \text{Tr}E)$, and note $0 \otimes 1 = \text{Choi matrix of some channel}$.

Note eventually, you can show that $\rho = \text{Tr}(C(0 \otimes 1))$.

Construly $\forall \rho \in D(X) \forall \omega$

- Note that $\forall (I_Y \otimes \rho) J(\omega) = \text{Tr}(\omega \otimes \rho^T)) = 1$. 

6 marks
(c) Super-trace-preserving part 2.

Let $\delta^*$ be the dual of $\delta$ defined in part 2.

Recall that $\delta^*(M) = \sum_i S_i^* E S_i$.

$\forall M \in L(Y) \otimes X'$

$\forall \xi \in D(X')$,

$\delta^*(1_y \otimes \xi) = 1_y \otimes N(\xi)$

$\exists \eta \in C(X', X)$ st.

Hint: Show that the following choice of $C$

$C = \delta^*(1_y \otimes \xi)$

satisfy the condition of $C$ in part 6, and apply 6.

Relate $\eta$ in the conclusion of part 6 to $N$ and $\xi$.

(d) Super-trace-preserving part 3.

Show that $(\delta^*) \Rightarrow \forall E \in L(Y \otimes X)$,

$\text{Tr}_{y} \delta^*(E) = N^*(\text{Tr}_{y} E)$

Hint: Show $\forall \xi \in D(X')$,

$\text{Tr}(\xi \cdot \text{Tr}_{y} \delta^*(E)) = \text{Tr}(\xi \cdot N^*(\text{Tr}_{y} E))$

from $X'$

Hint: Show that this equals to

$\text{Tr}_{y} \eta \cdot (1_y \otimes \xi) \cdot \delta(E)$
Note that part (i) shows that $\mathcal{S}$ takes trace-preserving $\Xi$ to trace-preserving $\mathcal{S}(\Xi)$.

To see this, recall $E \text{ TP } \iff \text{ trout } J(E) = 1$ in $\mathcal{S}$.

Now, try $\mathcal{S}(J(\Xi))$

$= N^* (\text{ try } J(\Xi))$ due to part (i)

$= N^* (I_X)$ if $\Xi$ is TP

$= I_X$.

So $\mathcal{S}(\Xi)$ is also TP.

No surprise! We assumed this in (i)

but the axiomatic assumption in (i) is now translated to concrete structural properties of $\mathcal{S}$ and $\mathcal{A}$!
(e) Structural theorem for $S$ or $A$.

From (b), $\forall E \in L(Y \otimes X), \text{try } x(E) = N^*(x \otimes y) E$

If the channel $N \in C(X',X)$ has Kraus rep

$$\forall M \in L(X'), \quad N(M) = \sum_l N^*_l M N_l \quad (N \in L(X',X))$$

then the dual has Kraus rep

$$\forall M \in L(X), \quad N^*(M) = \sum_l N^*_l M N_l \quad (N^* \in L(X',X'))$$

Remember $N^*$ is (in, CP, unital) (not necessarily TP).

Using the Kraus rep for $A$ in part (b)

and $\ldots \ldots N^*$ above, we have

$$\text{try } \sum_l \Sigma_l E S_i^* = \sum_l N^*_l \text{ (try } E \text{ ) } N_l$$

Take $\{1k^*\}$ to be o.n basis for $Y$'

$\{1k\}$ o.n $Y$.

$$\sum_l \sum_{k^*} (1k^* \otimes 1x) (\Sigma_l E S_i^* (1k) \otimes 1_k)$$

$$= \sum_l \sum_{k^*} N^*_l (1k^* \otimes 1x) E (1k) \otimes 1_k) N_l$$

$$= \sum_l \sum_{k^*} (1k^* \otimes N^*_l) E (1k) \otimes N_l$$

Sanity check: both sides in $L(X')$. 
Both sides are Kraus reps of a CP lin map from $L(Y \otimes X)$ to $L(X')$.

By a theorem in Nielsen & Chuang (likely in QC710, LN201) the Kraus operation on the two sides are related.

The Kraus rep with fewer Kraus ops is the canonical one.

So, by choosing $N(M') = \sum N_k M'_k N_k^*$ to be canonical (with fewest # of terms) and by choosing $|1_k>4$ to be on (thus, min #), the bottom line is canonical.

\[ \forall \text{ Isometry } W \text{ s.t.} \]
\[ (|k'1\otimes 1x')_i = \sum_{k,l} W_{k'i} W_{k,l} (|k1\otimes N_k^*) \]

with $W^*W = 1$.

If $i \in \Sigma_1$, let $A = C_{i1}$, if $i \in \Sigma_2$, let $B = C_{i2}^*$. Then $W_{k'i}, \forall k$ defines a matrix $W$ taking $1_k>1e$ to $1_{k'}1_i>$(see p8, Part 1, 1ec1, ye) s.t. $W \in L(Y \otimes B, Y' \otimes A)$.

\[ \forall \text{ s.t.} \]
\[ W_{k'i}, \forall k = \langle k'k|11W|1k>1e) \]
Sub (\$) into (\$):

\[ (k' l \otimes 1_{x'}) s_i = \sum_{k l i} (k', i l \otimes 1_k l' i') (k l \otimes N_{l}^*) \]

Show that the above implies

\[ s_i = \left( 1_y \otimes \langle i l_A \rangle W \right) \otimes 1_{x'} \cdot \left( 1_y \otimes \sum_{l} \langle l_\otimes N_{l}^* \rangle \right) \]

\[ \downarrow \quad \downarrow \quad \downarrow \]

\[ l(y \otimes', y \otimes x') \quad l(y \otimes B, y \otimes A) \quad l(x, B) \quad l(x, x') \]

So sub above to trans rep of \( A \):

\[ A(E) = \sum_{l} \left( 1_y \otimes \langle i l_A \rangle W \right) \otimes 1_{y'} \cdot (1_y \otimes Z) \cdot E \cdot (1_y \otimes Z^*) \cdot \left[ W^* (1_y \otimes l_A) \otimes 1_{x'} \right] \]

\[ = \text{tr}_A \left( W \otimes 1_{x'} \right) \otimes (1_y \otimes Z) \cdot E \cdot (1_y \otimes Z^*) \left( W^* \otimes 1_{x'} \right) \]

Let \( J(\otimes) = E \).
\[ V = \sum_{k} 1_{k} \otimes \bar{N}_k = \text{partial transpose of } \bar{Z}. \]
Let $\rho_0, \rho_1 \in \mathcal{D}(\mathcal{X})$ be density operators, for some complex Euclidean space $\mathcal{X}$, let $n$ be a positive integer, and define real numbers $a, \beta \in [0, 1]$ as

$$a = \frac{1}{2} \|\rho_0 - \rho_1\|_1 \quad \text{and} \quad \beta = F(\rho_0, \rho_1).$$

(a) Prove that

$$1 - \exp\left(-\frac{na^2}{2}\right) \leq \frac{1}{2} \|\rho_0 \otimes^n - \rho_1 \otimes^n\|_1 \leq na.$$

(b) For density operators $\sigma_0, \sigma_1 \in \mathcal{D}(\mathcal{X} \otimes^n)$ defined as

$$\sigma_0 = \frac{1}{2^{n-1}} \sum_{a_1, \ldots, a_n \in \{0, 1\}} \rho_{a_1} \otimes \cdots \otimes \rho_{a_n},$$

$$\sigma_1 = \frac{1}{2^{n-1}} \sum_{a_1, \ldots, a_n \in \{0, 1\}} \rho_{a_1} \otimes \cdots \otimes \rho_{a_n},$$

prove that

$$1 - n\beta \leq \frac{1}{2} \|\sigma_0 - \sigma_1\|_1 \leq \exp\left(-\frac{n\beta^2}{2}\right).$$

When answering both parts of this question, it may be helpful to make use of the fact that for every real number $\lambda \geq 1$, the inequality

$$\left(1 - \frac{1}{\lambda}\right)^\lambda < \frac{1}{e}$$

is satisfied.