On Degradable Quantum Channels

by Yingkai Ouyang.. Main Reference : quant-ph/0802.1360v2, The structure of degradable quantum channels, Cubitt, Ruskai, Smith

1 Complementary Channels

$$\Phi : M_{d_A} \to M_{d_B}.$$

$$\Phi(\rho) = \sum_k A_k \rho A_k^\dagger, \sum_k A_k^\dagger A_k = I_{d_A}.$$

Now define

$$W = \sum_k |k\rangle \otimes A_k.$$

Then

$$W\rho W^\dagger = \sum_{j,k} |j\rangle\langle k| \otimes A_j \rho A_k^\dagger.$$

$$Tr_E(W\rho W^\dagger) = \Phi(\rho)$$

For convention, define

$$\Phi^C(\rho) = Tr_A(W\rho W^\dagger).$$

'A' labels the system, 'E' labels the environment. **Choi rank**

$$d_E := rk\,(J(\Phi)) = rk\left(\sum_{i,j=0}^{d_A-1} |i\rangle\langle j| \otimes \Phi(|i\rangle\langle j|)\right)$$

is the minimal number of Kraus operators needed to represent Φ. One can check that

$$\Phi^C(\rho) = \sum_\mu R_\mu \rho R_\mu^\dagger$$

where

$$\langle j|R_\mu|k\rangle = \langle \mu|A_j|k\rangle,$$

$\mu \in \{0, ..., d_B - 1\}, k \in \{0, ..., d_A - 1\}$.

$$\boxed{\text{The } j\text{-th row of } R_\mu \text{ is the } \mu\text{-th row of } A_j.}$$

2 Degradable Channels

Definition: A channel is **degradable** if there exists a CPT Ψ such that

$$\Psi \circ \Phi = \Phi^C,$$

that is

$$\Psi(\Phi(\rho)) = \Phi^C(\rho) \quad \forall \rho \in M_{d_A}$$

Fact: Φ degradable $\implies$ ker $\Phi \subseteq$ ker Φ^C.

Easy to show Facts:

- $d_A = 1 \implies \Phi, \Phi^C$ degradable, anti-degradable
- $d_B = 1 \implies \Phi = Tr \implies \Phi$ antidegradable
- $d_E = 1 \implies \Phi(\rho) = U\rho U^\dagger, U^\dagger U = I_{d_A} \implies \Phi^C = Tr \implies \Phi$ degradable, $\Phi = Tr$

Thm 1: Suppose $\Phi : M_{d_A} \to M_{d_B}$ maps every pure state to a pure state. Then either

1. $d_A \le d_B$ and $\Phi(\rho) = U\rho U^\dagger$, $U^\dagger U = I_{d_A}$, Φ degradable, $d_E = 1$.
2. $\Phi(\rho) = Tr(\rho)|\phi\rangle\langle\phi|, \Phi$ antidegradable.

holds

Thm3 Let Φ be CPT such that there exists $|\psi\rangle \in \mathbb{C}^{d_A}$ with $rank(\Phi(|\psi\rangle\langle\psi|)) = d_B$. Then if Φ is degradable, then $d_E = d_B$.

Thm4: Let $\Phi : M_{d_A} \to M_2$ be a CPT map with **qubit output**. If Φ is degradable, then (i) $d_E \le 2$, (ii) and $d_A \le 3$.

Proof of Thm4:

(i) If $maxrank_\rho(\Phi(\rho)) = 1$, then by **Thm1**, $d_E = 1$.
If $maxrank_\rho(\Phi(\rho)) = 2$, then by **Thm3**, $d_E = d_B = 2$.

(ii) $d_E \le 2 \implies \Phi(\rho) = A\rho A^\dagger + B\rho B^\dagger$. For $a_1, a_2 \in [0, 1]$,

$$A = \begin{pmatrix} \sqrt{a_1} & 0 & 0...0 \\ 0 & \sqrt{a_2} & 0...0 \end{pmatrix}$$

and

$$B^\dagger B = I_{d_A} - A^\dagger A = diag(1 - a_1, 1 - a_2, 1, ..., 1)$$

But B is $2 \times d_A$ matrix $\implies rk(B) \le 2 \implies rk(B^\dagger B) \le 2 \implies d_A \le 4$.

If $d_A = 4$, then $a_1 = a_2 = 1$. But ker Φ is not contained in ker Φ^C which is a contradiction. Hence $d_A \le 3$. **end of Thm4** ∎

Thm by Wolf, Perez-Garcia Let $\Phi : M_2 \to M_2$ have Choi rank 2. If Φ is degradable or antidegradable, its Kraus operators are

$$A_0 = \begin{pmatrix} \cos\alpha & 0 \\ 0 & \cos\beta \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & \sin\alpha \\ \sin\beta & 0 \end{pmatrix}. \tag{2.1}$$

Significance of Thm4

Thm10: Let Φ have qutrit output. If Φ degradable, $d_E \leq 3$

Question: What about results with $d_B = 4, 5, 6, ...$? $d_E \leq d_B$?

Answer: No. If $d_B = 2d_A$ then we can have $d_E > d_B$ (Construction in reference). CRS also construct channels with $d_A = d_B = 6d$, and $d_E = 3(d^2+1) > 6d = d_B$. But what about $d_B = 4, 5$.

3 What other channels are degradable?

Thm11 Every channel with rank 1 Kraus operators is anti-degradable.

Proof is constructive.

Many more examples of antidegradable channels given. LOTS of such channels. (although it is remarked that most channels are neither degradable/ antidegradable)

4 Applications

The qubit amplitude damping channel (degradable) has been used by SSW to improve the upper-bound for the depolarization channel. If $\mathcal{N}, \mathcal{M}$ are degradable, then

$$Q(\lambda\mathcal{N} + (1-\lambda)\mathcal{M}) \leq \lambda Q(\mathcal{N}) + (1-\lambda)Q(\mathcal{M}).$$

Quantum capacity of **degradable** channels can be efficiently evaluated because of several results.

- I_{coh} is additive for Φ degradable.
- $I_{coh}(\Phi, \rho)$ is concave function of ρ for Φ degradable, implies that we only need to consider diagonal ρ.

- $I_{coh}(\Phi, \rho) = S(\Phi(\rho)) - S(\Phi^C(\rho))$

$$\frac{1}{2}A_\gamma(\rho) + \frac{1}{2}XA_\gamma(\rho)X = \mathcal{N}_\gamma(\rho)$$

where $\mathcal{N}_\gamma$ has Kraus operators

$$\sqrt{p_x}X, \sqrt{p_y}Y, \sqrt{p_z}Z, \sqrt{1 - p_x - p_y - p_z}I$$

and

$$p_x = p_y = \frac{\gamma}{4}, p_z = \frac{1 - \frac{\gamma}{2} - \sqrt{1-\gamma}}{2}$$

Now define $H = \frac{X+Z}{2}$, $H_{yz} = \frac{Y+Z}{2}, H_{xy} = \frac{X+Y}{2}$. Conjugation of a nontrivial Pauli by H takes $X \to Z, Y \to Y, Z \to X$. Conjugation of a nontrivial Pauli by H_{yz} takes $X \to X, Y \to Z, Z \to Y$. Conjugation of a nontrivial Pauli by H_{xy} takes $X \to Y, Y \to X, Z \to Z$. Now let Φ_p be a quantum channel such that

$$\Phi_p(\rho) = \frac{\mathcal{N}_\gamma(\rho) + H\mathcal{N}_\gamma(H^\dagger \rho H)H^\dagger + H_{yz}\mathcal{N}_\gamma(H_{yz}^\dagger \rho H_{yz})H_{yz}^\dagger}{3}.$$

Then Φ_p is a depolarization channel of noise parameter $p = (p_x + p_y + p_z)/3$.

5 Further questions

Can we use a dimension 2^m dimension amplitude damping channel to also obtain an upper bound for the quantum capacity of the depolarization channel? Tensor product of m qubit amplitude damping channels is not equal to a 2^m dimension amplitude damping channel in general.

Let $\vec{\gamma} = (\gamma_1, ..., \gamma_{d-1})$ It turns out that the following channel $\mathcal{A}_{\vec{\gamma}}^{(d)} : M_d \to M_d$, with Kraus operators

$$A_0 = |0\rangle\langle 0| + \sum_{i=1}^{d-1} \sqrt{1-\gamma_i}|i\rangle\langle i| \tag{5.1}$$

$$A_i = \sqrt{\gamma_i}|0\rangle\langle i|, \quad i \in [d-1] \tag{5.2}$$

for real $\gamma_i \in [0, 1]$. It follows that the complementary channel for $\mathcal{A}^{(d)}$ have the Kraus operators

$$R_0 = |0\rangle\langle 0| + \sum_{i=1}^{d-1} \sqrt{\gamma_i}|i\rangle\langle i| \tag{5.3}$$

$$R_i = \sqrt{1-\gamma_i}|0\rangle\langle i|, \quad i \in [d-1] \tag{5.4}$$

Now let $\vec{\lambda} = (\lambda_1, ..., \lambda_{d-1})$ such that $\lambda_i = \frac{1-2\gamma_i}{1-\gamma_i}$. If $0 \le \gamma_i \le \frac{1}{2}$ for all $i \in [d-1]$, then $\mathcal{A}_{\vec{\lambda}}^{(d)}$ is a well defined CPT channel and $\mathcal{A}_{\vec{\gamma}}^{(d)} \circ \mathcal{A}_{\vec{\lambda}}^{(d)} = \mathcal{A}_{\mathbf{1}-\vec{\gamma}}^{(d)}$. Thus $\mathcal{A}_{\vec{\gamma}}^{(d)}$ is degradable if $0 \le \gamma_i \le \frac{1}{2}$ for all $i \in [d-1]$.

References

[1] M. M. Wolf and D. Perez-Garcia, "Quantum capacities of channels with small environment," *Phys. Rev. A*, vol. 75, no. 012303, 2007.

[2] T. S. Cubitt, M. B. Ruskai, and G. Smith, "The structure of degradable quantum channels," *Journal of Mathematical Physics*, vol. 49, no. 10, 2008.