On Degradable Quantum Channels

1 Complementary Channels

\[ \Phi : M_{d_A} \rightarrow M_{d_B} \]
\[ \Phi(\rho) = \sum_k A_k \rho A_k^\dagger, \sum_k A_k^\dagger A_k = I_{d_A}. \]

Now define
\[ W = \sum_k |k\rangle \otimes A_k. \]

Then
\[ W\rho W^\dagger = \sum_{j,k} |j\rangle \langle k| \otimes A_j \rho A_k^\dagger. \]
\[ Tr_E(W\rho W^\dagger) = \Phi(\rho) \]

For convention, define
\[ \Phi^C(\rho) = Tr_A(W\rho W^\dagger). \]

‘A’ labels the system, ‘E’ labels the environment. Choi rank
\[ d_E := rk(J(\Phi)) = rk\left( \sum_{i,j=0}^{d_A-1} |i\rangle \langle j| \otimes \Phi(|i\rangle \langle j|) \right) \]
is the minimal number of Kraus operators needed to represent \( \Phi \).

One can check that
\[ \Phi^C(\rho) = \sum_{\mu} R_{\mu} \rho R_{\mu}^\dagger \]
where
\[ \langle j | R_\mu | k \rangle = \langle \mu | A_j | k \rangle, \]
\( \mu \in \{0, ..., d_B - 1\}, k \in \{0, ..., d_A - 1\}. \)

The \( j \)-th row of \( R_\mu \) is the \( \mu \)-th row of \( A_j \).

2 Degradable Channels

**Definition:** A channel is **degradable** if there exists a CPT \( \Psi \) such that
\[ \Psi \circ \Phi = \Phi^C, \]
that is
\[ \Psi(\Phi(\rho)) = \Phi^C(\rho) \quad \forall \rho \in M_{d_A} \]

**Fact:** \( \Phi \) degradable \( \implies \) ker \( \Phi \subseteq \ker \Phi^C. \)

**Easy to show Facts:**

- \( d_A = 1 \implies \Phi, \Phi^C \) degradable, anti-degradable
- \( d_B = 1 \implies \Phi = Tr \implies \Phi \) antidegradable
- \( d_E = 1 \implies \Phi(\rho) = U\rho U^\dagger, U^\dagger U = I_{d_A} \implies \Phi^C = Tr \implies \Phi \) degradable, \( \Phi = Tr \)

**Thm 1:** Suppose \( \Phi : M_{d_A} \rightarrow M_{d_B} \) maps every pure state to a pure state. Then either
1. $d_A \leq d_B$ and $\Phi(\rho) = U\rho U^\dagger$, $U^\dagger U = I_{d_A}$, $\Phi$ degradable, $d_E = 1$.

2. $\Phi(\rho) = Tr(\rho)|\phi\rangle\langle \phi|$, $\Phi$ antidegradable.

holds

**Thm3** Let $\Phi$ be CPT such that there exists $|\psi\rangle \in \mathbb{C}^{d_A}$ with $\text{rank}(\Phi(|\psi\rangle\langle \psi|)) = d_B$. Then if $\Phi$ is degradable, then $d_E = d_B$.

**Thm4**: Let $\Phi : M_{d_A} \to M_2$ be a CPT map with **qubit output**. If $\Phi$ is degradable, then (i) $d_E \leq 2$, (ii) and $d_A \leq 3$.

**Proof of Thm4:**

(i) If $\text{maxrank}_\rho(\Phi(\rho)) = 1$, then by **Thm1**, $d_E = 1$.

If $\text{maxrank}_\rho(\Phi(\rho)) = 2$, then by **Thm3**, $d_E = d_B = 2$.

(ii) $d_E \leq 2 \implies \Phi(\rho) = A\rho A^\dagger + B\rho B^\dagger$. For $a_1, a_2 \in [0, 1]$,

$$A = \begin{pmatrix} \sqrt{a_1} & 0 & 0 & \ldots & 0 \\ 0 & \sqrt{a_2} & 0 & \ldots & 0 \end{pmatrix}$$

and

$$B^\dagger B = I_{d_A} - A^\dagger A = \text{diag}(1 - a_1, 1 - a_2, 1, \ldots, 1)$$

But $B$ is $2 \times d_A$ matrix $\implies \text{rk}(B) \leq 2 \implies \text{rk}(B^\dagger B) \leq 2 \implies d_A \leq 4$.

If $d_A = 4$, then $a_1 = a_2 = 1$. But $\ker \Phi$ is not contained in $\ker \Phi^C$ which is a contradiction. Hence $d_A \leq 3$. **end of Thm4**

**Thm by Wolf, Perez-Garcia** Let $\Phi : M_2 \to M_2$ have Choi rank 2. If $\Phi$ is degradable or antidegradable, its Kraus operators
are
\[ A_0 = \begin{pmatrix} \cos \alpha & 0 \\ 0 & \cos \beta \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & \sin \alpha \\ \sin \beta & 0 \end{pmatrix}. \] (2.1)

**Significance of Thm4**

**Thm10:** Let \( \Phi \) have qutrit output. If \( \Phi \) degradable, \( d_E \leq 3 \)

**Question:** What about results with \( d_B = 4, 5, 6, \ldots \)? \( d_E \leq d_B \)?
**Answer:** No. If $d_B = 2d_A$ then we can have $d_E > d_B$ (Construction in reference). CRS also construct channels with $d_A = d_B = 6d$, and $d_E = 3(d^2 + 1) > 6d = d_B$. But what about $d_B = 4, 5$.

3 What other channels are degradable?

**Thm11** Every channel with rank 1 Kraus operators is anti-degradable.

**Proof is constructive.**

Many more examples of antidegradable channels given. LOTS of such channels. (although it is remarked that most channels are neither degradable/ antidegradable)

4 Applications

The qubit amplitude damping channel (degradable) has been used by SSW to improve the upper-bound for the depolarization channel. If $\mathcal{N}, \mathcal{M}$ are degradable, then

$$Q(\lambda \mathcal{N} + (1 - \lambda)\mathcal{M}) \leq \lambda Q(\mathcal{N}) + (1 - \lambda)Q(\mathcal{M}).$$

Quantum capacity of **degradable** channels can be efficiently evaluated because of several results.

- $I_{coh}$ is additive for $\Phi$ degradable.
- $I_{coh}(\Phi, \rho)$ is concave function of $\rho$ for $\Phi$ degradable, implies that we only need to consider diagonal $\rho$. 

5
\[ I_{coh}(\Phi, \rho) = S(\Phi(\rho)) - S(\Phi^C(\rho)) \]
\[
\frac{1}{2} A_\gamma(\rho) + \frac{1}{2} X A_\gamma(\rho) X = N_{\gamma}(\rho)
\]
where \( N_{\gamma} \) has Kraus operators
\[
\sqrt{p_x} X, \sqrt{p_y} Y, \sqrt{p_z} Z, \sqrt{1 - p_x - p_y - p_z} I
\]
and
\[
p_x = p_y = \frac{\gamma}{4}, p_z = \frac{1 - \frac{\gamma}{2} - \sqrt{1 - \gamma}}{2}
\]
Now define \( H = \frac{X+Z}{2}, H_{yz} = \frac{Y+Z}{2}, H_{xy} = \frac{X+Y}{2} \). Conjugation of a nontrivial Pauli by \( H \) takes \( X \rightarrow Z, Y \rightarrow Y, Z \rightarrow X \). Conjugation of a nontrivial Pauli by \( H_{yz} \) takes \( X \rightarrow X, Y \rightarrow Z, Z \rightarrow Y \). Conjugation of a nontrivial Pauli by \( H_{xy} \) takes \( X \rightarrow Y, Y \rightarrow X, Z \rightarrow Z \). Now let \( \Phi_p \) be a quantum channel such that
\[
\Phi_p(\rho) = \frac{N_{\gamma}(\rho) + H N_{\gamma}(H^{\dagger} \rho H) H^{\dagger} + H_{yz} N_{\gamma}(H^{\dagger}_{yz} \rho H_{yz}) H_{yz}^{\dagger}}{3}.
\]
Then \( \Phi_p \) is a depolarization channel of noise parameter \( p = (p_x + p_y + p_z)/3 \).

5 Further questions

Can we use a dimension \( 2^m \) dimension amplitude damping channel to also obtain an upper bound for the quantum capacity of the depolarization channel? Tensor product of \( m \) qubit amplitude damping channels is not equal to a \( 2^m \) dimension amplitude damping channel in general.
Let $\vec{\gamma} = (\gamma_1, ..., \gamma_{d-1})$. It turns out that the following channel $A_{\vec{\gamma}}^{(d)} : M_d \rightarrow M_d$, with Kraus operators

\begin{align}
A_0 &= |0\rangle\langle 0| + \sum_{i=1}^{d-1} \sqrt{1 - \gamma_i} |i\rangle\langle i| \\
A_i &= \sqrt{\gamma_i} |0\rangle\langle i|, \quad i \in [d - 1]
\end{align}

(5.1) (5.2)

for real $\gamma_i \in [0, 1]$. It follows that the complementary channel for $A^{(d)}$ have the Kraus operators

\begin{align}
R_0 &= |0\rangle\langle 0| + \sum_{i=1}^{d-1} \sqrt{\gamma_i} |i\rangle\langle i| \\
R_i &= \sqrt{1 - \gamma_i} |0\rangle\langle i|, \quad i \in [d - 1]
\end{align}

(5.3) (5.4)

Now let $\vec{\lambda} = (\lambda_1, ..., \lambda_{d-1})$ such that $\lambda_i = \frac{1 - 2\gamma_i}{1 - \gamma_i}$. If $0 \leq \gamma_i \leq \frac{1}{2}$ for all $i \in [d - 1]$, then $A_{\vec{\lambda}}^{(d)}$ is a well defined CPT channel and $A_{\vec{\gamma}}^{(d)} \circ A_{\vec{\lambda}}^{(d)} = A_{1 - \vec{\gamma}}^{(d)}$. Thus $A_{\vec{\gamma}}^{(d)}$ is degradable if $0 \leq \gamma_i \leq \frac{1}{2}$ for all $i \in [d - 1]$.

References
