CO 781 Assignment 2, Spring 2010, due June 29th in class.

Question 1. On accessible information [20 marks]

(a) Superadditivity:

Let the ensemble $\mathcal{E}$ be: $|v_1\rangle = |0\rangle$, $|v_2\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$, $|v_3\rangle = \frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle$ and $p_1 = p_2 = p_3 = \frac{1}{3}$.

Recall from class that $I_{acc}(\mathcal{E}) = 0.5850$, attained by the optimal measurement with POVM elements $M_i = \frac{2}{3}(I - |v_i\rangle\langle v_i|)$ for $i = 1, 2, 3$.

(i) What is the pretty good measurement (PGM) for $\mathcal{E}$?

(ii) How much mutual information is given by the PGM?

(iii) If 2 copies of the states are given, what is the new PGM? (i.e., $\forall_i, |v_i\rangle^{\otimes 2}$ is given with probability 1/3.)

(iv) How much mutual information can be extracted, and how is it compared to $2I_{acc}(\mathcal{E})$?

(b) Additivity: For 2 ensembles $\mathcal{E}_1 = \{p_i, \rho_i\}$ and $\mathcal{E}_2 = \{q_j, \eta_j\}$, let $\mathcal{E}_1 \otimes \mathcal{E}_2 = \{p_i q_j, \rho_i \otimes \eta_j\}$. Prove that $I_{acc}(\mathcal{E}_1 \otimes \mathcal{E}_2) = I_{acc}(\mathcal{E}_1) + I_{acc}(\mathcal{E}_2)$. (Try without hint first ... and read 0103098 VII as a first hint.)

(c) Why (a),(b) are not contradicting one another?

Question 2. The tetrahedron states [20 marks]

Let I, X, Y, Z, denote the Pauli matrices. Consider 4 states in $\mathbb{C}^2$ given by:

$|\psi_1\rangle\langle\psi_1| = \frac{1}{2}(I + \frac{1}{\sqrt{3}}(X + Y + Z))$

$|\psi_2\rangle\langle\psi_2| = \frac{1}{2}(I + \frac{1}{\sqrt{3}}(X - Y - Z))$

$|\psi_3\rangle\langle\psi_3| = \frac{1}{2}(I + \frac{1}{\sqrt{3}}(-X + Y - Z))$

$|\psi_4\rangle\langle\psi_4| = \frac{1}{2}(I + \frac{1}{\sqrt{3}}(-X - Y + Z))$

Note that $|\langle\psi_i|\psi_j\rangle|$ is constant for $i \neq j$, and the Bloch vectors of these states form the vertices of a tetrahedron.

(a) What is the pretty good measurement (PGM) corresponding to these states?

(b) For the ensemble $\mathcal{E}$ in which each $|\psi_i\rangle$ is drawn with probability 1/4, what is the probability of failure in the PGM?

(c) Does the PGM minimize the probability of error? Does it attains $I_{acc}(\mathcal{E})$?

(d) What is the classical capacity of a Q-box capable of emitting $|\psi_i\rangle$ $(i = 1, 2, 3, 4)$?

(e) Let $\mathcal{N}$ be the channel that measures the input ρ with the PGM in (b), and upon the outcome j, outputs the state $|\psi_j\rangle$ to Bob. What are the optimal ensemble, $\chi^{(1)}(\mathcal{N})$, and $C(\mathcal{N})$? (Hint: $\mathcal{N}$ turns out a bit too special, so, a short cut to this question is to understand what $\mathcal{N}$ does.)