Quantum Error Correction and Fault Tolerance, Winter 2022

Part II: Fault Tolerance

Problem Set 2
Due: Friday March 4 2022 10pm.

1. State injection circuits [5 marks]

The control-$S$ gate is in the third level of the Clifford hierarchy and is diagonal in the $Z$ basis. Recall that $S = \text{diag}(1, i)$ and that control-$S$, $CS = \text{diag}(1, 1, 1, i)$. We can derive a state injection circuit for $CS$ starting from the standard teleportation circuit shown in Figure 1.

![Figure 1: Teleportation circuit](image1)

(a) Show that $CS^\dagger (X \otimes X) CS = XS \otimes S^\dagger X$. [2 marks]

(b) Commute the $CS$ gate in Figure 2 backwards through the circuit until the input ancilla state is the magic state $CS|+\rangle$ and write down the resultant state injection circuit. [3 marks]

![Figure 2: Teleportation circuit with CS](image2)

2. Transversal gates of the 5-qubit code [8 marks]

Recall that the stabilizer of the 5-qubit code is generated by the cyclic permutations of $XZZXI$. The logical operators are $\overline{X} = X^{\otimes 5}$ and $\overline{Z} = Z^{\otimes 5}$

(a) Show that $Y^{\otimes 5}$ is an implementation of logical $Y$ in the 5-qubit code. [1 mark]
(b) Compute the transformation of $X$, $Y$ and $Z$ under conjugation by the Clifford gate $K = \exp(\frac{2\pi}{3\sqrt{3}}(X + Y + Z))$. [4 marks]

Hint: use the identity $\exp(i\theta \vec{v} \cdot \vec{σ}) = \cos \theta I + \sin \theta \vec{v} \cdot \vec{σ}$, where $\vec{σ} = (X,Y,Z)^T$ and $||\vec{v}|| = 1$.

(c) Show that $\overline{K} = K^\otimes 5$ in the 5-qubit code, i.e., this gate is transversal. [3 marks]

3. Transversal gates of the $[[8,3,2]]$ code [14 marks]

The $[[8,3,2]]$ code is the smallest known stabilizer code with a transversal non-Clifford gate. Its stabilizer group is

$\langle X^\otimes 8, Z_1Z_2Z_3Z_4, Z_5Z_6Z_7Z_8, Z_1Z_2Z_5Z_6, Z_2Z_4Z_6Z_8 \rangle$.

An easy way to understand the code is to consider a cube with qubits placed at the vertices, as shown in Figure 3. In this picture, the $Z$ stabilizer generators are associated with faces, i.e., for each face $f$ we have the stabilizer $Z(f) = \prod_{v \in f} Z_v$, where $Z_v$ denotes $Z$ applied to the qubit at vertex $v$. In this picture we can also define a convenient basis for the logical operators where the logical $X$ operators are associated with faces and the logical $Z$ operators are associated with edges. That is, $X_1 = X_1X_2X_3X_4$, $X_2 = X_1X_2X_5X_6$, $X_3 = X_2X_4X_5X_8$, $Z_1 = Z_1Z_5$, $Z_2 = Z_1Z_3$ and $Z_3 = Z_1Z_2$.

Figure 3: $[[8,3,2]]$ code qubit labels

(a) Write down the encoded computational basis states of the $[[8,3,2]]$ code. [4 marks]

(b) Show that $S_2S_4^\dagger S_6^\dagger S_6$ implements a logical $\overline{CZ}_{12}$ gate (acts as $CZ$ on encoded qubits 1 and 2). [4 marks]

Hint: you can either apply the gate to the encoded computational basis states or conjugate the logical Pauli operators. Recall that $CZ|xy\rangle = (-1)^{xy}|xy\rangle$ for $x, y \in \{0,1\}$ and $CZ(X \otimes I)CZ^\dagger = X \otimes Z$.

(c) Write down transversal implementations of $\overline{CZ}_{14}$ and $\overline{CZ}_{23}$. [2 marks]

(d) Show that $T_1T_2^\dagger T_3^\dagger T_4T_5^\dagger T_6T_7^\dagger$ implements a logical $\overline{CCZ}$ gate. [4 marks]

Hint: Recall that $CCZ|xyz\rangle = (-1)^{xyz}|xyz\rangle x, y, z \in \{0,1\}$ and $CCZ(X \otimes I \otimes I)CCZ^\dagger = X \otimes CZ$. 

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