MATH 135 Assignment 2, Winter 2010

Due 8:30am on Wednesday January 20, 2010 in the drop boxes opposite the Tutorial Centre, MC 4066.

Instructions: First attempt questions 1-2 and use page 2 to check your answers (do not turn these in). **Hand in the solutions for questions 3-6 by the due date (total 38 marks).** Question 7 is for extra credit, due the same time as the rest. A solution to an LDE is positive if the solutions to x and y are positive simultaneously, and nonnegative if they're both nonnegative. Additional practice problems are 39, 43, 53, 57, 63, 65 on p51 of the textbook.

Question 1. Practice question on the EEA and the gcd characterization theorem [0 marks]

Prove that $\gcd(ab, c) = 1$ if and only if $\gcd(a, c) = \gcd(b, c) = 1$.

Question 2. Practice questions on linear Diophantine equations [0 marks]

For each linear Diophantine equation, explain whether there is a solution or not. Write down a complete solution if one exists.

(a) $14x + 91y = 200$

(b) $331x + 249y = 1$

(c) $331x - 249y = 3$

Question 3. Linear Diophantine equation with constraints [Parts (a) and (b) carry 6 and 8 marks respectively, 14 marks total].

(a) Determine all solutions to the linear Diophantine equation $57x + 135y = 8400$.

(b) Find all the nonnegative solutions of the above that also satisfy the inequality $x + 2y \leq 140$.

Question 4. Application of linear Diophantine equations [5 marks]

A store had to deliver boxes of cheesecake slices, a dozen in each box. A bakery supplied a number of cakes to the store, but each was cut (by mistake) into 11 slices instead of 12. The storekeeper thus repackaged the slices into boxes of 12, and had 3 left over.

Find the minimum possible number of cakes delivered to the store, by rephrasing the question as an LDE and proving the correctness of the solution (instead of enumerating possibilities).

Question 5. Further questions for linear Diophantine equations

[Parts (a) and (b) carry 10 and 5 marks respectively, 15 marks total]

(a) Suppose a, b not both zero. List all a, b for which $ax + by = c$ have infinitely many positive integer solutions (in terms of c). Explain.

(b) If $\gcd(a, b) = 1$ and $ax+by = c$ has a strictly positive integer solution, is it always true that $ax+by = d$ also has a strictly positive integer solution whenever $d > c$? Why?

Question 6. Integers in different bases [4 marks, 2 marks for each part]

(a) Convert $(1234)_{10}$ to base 3.
(b) Convert $(5163)_8$ to base 10.

Question 7. Challenge question for linear Diophantine equation [10 marks extra credit]

For what values of c does $2x + 5y = c$ have exactly one *strictly* positive integer solution? (A little enumeration is needed in the final stage to obtain the answer.)

Question 1. Practice question on the EEA and the gcd characterization theorem

Prove that $\gcd(ab, c) = 1$ if and only if $\gcd(a, c) = \gcd(b, c) = 1$.

Proof:

If $\gcd(ab, c) = 1$:

- then, by the EEA, $\exists x, y \in \mathbb{Z}$ such that $abx + cy = 1$.
- Rewriting the above as $a(bx) + cy = 1$ and applying the gcd characterization theorem with $d = 1$, we conclude that $\gcd(a, c) = 1$.
- Similarly, the alternative rewriting as $b(ax) + cy = 1$ gives $\gcd(b, c) = 1$.

If $\gcd(a, c) = \gcd(b, c) = 1$:

- then, by the EEA, $\exists x, y, s, t \in \mathbb{Z}$ such that $ax + cy = 1$ and $bs + ct = 1$.
- Multiplying the above two expressions together and rearranging terms, we get $ab(xs) + c(ybs + axt + cyt) = 1$.
- Since $xs \in \mathbb{Z}$ and $ybs + axt + cyt \in \mathbb{Z}$, the gcd characterization theorem with $d = 1$ applies and we can conclude that $\gcd(ab, c) = 1$.

Comment: Have you put in the justifications (EEA and the gcd char thm) where they are used, and checked their assumptions each time you apply them?

Question 2. Practice on linear Diophantine equations

For each linear Diophantine equation, explain whether there is a solution or not. Write down a complete solution if one exists.

(a) $14x + 91y = 200$

Answer: Since $\gcd(14, 91) = 7$ and $7 \nmid 200$, LDE has no solution.

(b) $331x + 249y = 1$

Answer: Applying EEA:

1	0	331		
0	1	249		
1	-1	82	1	$331 = 249 \times 1 + 82$
-3	4	3	3	$249 = 82 \times 3 + 3$
82	-109	1	27	$82 = 3 \times 27 + 1$
		0	3	$3 = 1 \times 3 + 0$

we know that $\gcd(331, 249) = 1$ and $1|1$.

So there are solutions to the LDE, and $x_0 = 82$, $y_0 = -109$ is a particular solution.

The complete solution is given by $x = 82 + 249n$ and $y = -109 - 331n$ $\forall n \in \mathbb{Z}$.

Comment: Check if you have the $-$ sign in the n-term of the complete solution for y, and the quantifier for n.

(c) $331x - 249y = 3$

Answer:

- From (b), $331(82) + 249(-109) = 1$, so, $331(246) - 249(327) = 3$.
- Thus, $x_0 = 246$, $y_0 = 327$ is a particular solution.
- The complete solution is therefore $x = 246 - 249n$ and $y = 327 - 331n$ $\forall n \in \mathbb{Z}$.

Comments: The particular solution of (c) can be taken to be 3 times that of (b), **but multiplying the complete solution of (b) by 3 only gives one-out-of-three possible solutions for (c) – an incorrect answer**. Also check that you have the correct signs in the n-terms (both $-$ or both $+$).