Question 1. Alternative form of necessary and sufficient condition for QECC [4 marks]

You learnt from class that a codespace $C$ with projector $P$ is a QECC for the error set $E$ if and only if

$\forall E_i, E_j \in E, PE_i^\dagger E_j P = c_{ij} P$ for some $c_{ij} \in \mathbb{C}$.

Provide a brief argument that the codespace $C$ is a QECC for the error set $E$ if and only if there exists an orthonormal basis $\{|\psi_a\rangle\}$ for $C$, $\forall E_i, E_j \in E, \langle \psi_a | E_i^\dagger E_j | \psi_b \rangle = c_{ij} \delta_{a,b}$ for some $c_{ij} \in \mathbb{C}$, and $\delta$ denotes the Kronecker delta function.

Note that we can replace “there exists an” by “for any” and obtain another equivalent statement.

Question 2. Bosonic code for amplitude damping [10 marks]

Consider an infinite-dimensional Hilbert space with a basis $\{|0\rangle, |1\rangle, |2\rangle, |3\rangle, \cdots \}$ where $|j\rangle$ denotes a state with $j$ excitations (e.g., $j$ photons). Consider the amplitude damping channel $A_\gamma(\rho) = \sum_k A_k \rho A_k^\dagger$ where

$A_k = \sum_{j \geq k} \sqrt{\frac{j}{k}} \sqrt{(1 - \gamma)^{j-k} \gamma^k} |j-k\rangle\langle j|$

represents the loss of $k$ excitations from the system. In particular,

$A_0 = \sum_j (1 - \gamma) \frac{j}{2} |j\rangle\langle j|, \quad A_1 = \sum_{j \geq 1} \sqrt{j} (1 - \gamma)^{j-1} \gamma |j-1\rangle\langle j|.$

(a) [6 marks] Show that the codespace with basis

$|\psi_0\rangle = \frac{1}{\sqrt{2}}(|40\rangle + |04\rangle), \quad |\psi_1\rangle = |22\rangle$

is a QECC for the error set $E = \{A_0 \otimes A_0, A_0 \otimes A_1, A_1 \otimes A_0\}$.

(b) [4 marks] Describe a valid decoding operation for this QECC.

Question 3. Approximate error correction [6 marks]

Consider the QECC $C'$ with basis

$|\psi_0\rangle = \frac{1}{\sqrt{2}}(|4\rangle + |0\rangle), \quad |\psi_1\rangle = |2\rangle$

and the error set $E' = \{A_0, A_1\}$. Note that $C'$ is not a QECC for $E'$; provide a decoding operation $D$ so that $\|D(A_\gamma(\rho)) - \rho_1 \|_1 \approx O(\gamma^2)$.

PS. In the same way, any distance-3 qubit QECC also corrects for 1 amplitude damping error (requiring a blocklength of at least 5). A 4-qubit approximate QECC (in quant-ph/9704002) corrects for 1 amplitude damping error and the codespace is spanned by $\frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)^\otimes 2$. The similar Shor code corrects for 2 amplitude damping errors (Section 8.7 of Gottesman thesis)!