## Continuity of channel capacities

0810.4931
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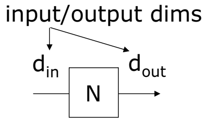


A capacity of a channel (e.g. C(N)) is a function taking each N to a real number.

If two channels are close to one another under some distance measure, should their capacities be similar?

## Continuity -- isn't it obvious ?

- Classical capacity of a classical channel
  Yes, expression is convex and single-letterized (with compact domain) in the input distribution
- Capacities of a quantum channel
  Many only have expression as an optimization over unbounded number of channel uses
  Even if $N \approx M$, $N^{\otimes n}$ & $M^{\otimes n}$ are very different

*Consider classical messages first ...*

Shannon's noisy coding theorem
$C(N) = \max_{p(x)} I(X:Y) = \max_{p(x)} I(X:N(X))$

HSW Theorem:
$C(N) = \lim_{n\to\infty} \max_{p_x,\rho_x} 1/n \; S(X:B_1B_2 \cdots B_n)$

eval on $\underbrace{\sum_x p_x|x\rangle\langle x|}_{X} \otimes \underbrace{N^{\otimes n}(\rho_x)}_{B_1B_2\cdots B_n}$

## Continuity of C(N) for classical channels:

$C(N) = \max_X \boxed{H(X)+H(N(X))-H(XN(X))} = \max_X f(X,N)$

For 2 channels $N_1$ and $N_2$,

the difference between $C(N_1)$ & $C(N_2)$ is caused by
- the difference between $N_1$ & $N_2$,
- also that between the optimal $X_1$ & $X_2$

We first remove this problem ...

## Continuity of C(N) for classical channels:

$C(N) = \max_X \boxed{H(X)+H(N(X))-H(XN(X))} = \max_X f(X,N)$

For 2 channels $N_1$ and $N_2$,

Let $X_i^{op}$ be optimal input distribution for $N_i$:

$\therefore C(N_1) \le C(N_2)$ + approx

$C(N_1) = f(X_1^{op},N_1)$ ←— $\approx$ equal —→ $f(X_1^{op},N_2)$

$\ge$ $\qquad\qquad$ $\le$

$f(X_2^{op},N_1)$ ←— if $\forall X$ f(X,N) cts in N these are $\approx$ equal —→ $f(X_2^{op},N_2) = C(N_2)$

$\therefore C(N_1) \ge C(N_2)$ – approx

$\therefore \; | C(N_1) - C(N_2) | \le \max_X | f(X,N_1) - f(X,N_2) |$

## Continuity of C(N) for classical channels:

$C(N) = \max_X \boxed{H(X)+H(N(X))-H(XN(X))} = \max_X f(X,N)$

$| C(N_1) - C(N_2) | \le \max_X | f(X,N_1) - f(X,N_2)|$

When does $N_1 \approx N_2$ imply $\forall X$ $|f(X,N_1) - f(X,N_2)|$ small?

(a) Want $N_1 \approx N_2$ implies $\forall X$ $XN_1(X) \approx XN_2(X)$
Take $||N_1-N_2|| = \max_X ||XN_1(X)-XN_2(X)||_{tr}$

(b) Want f(X,N) is smooth in N:
$\Delta f \le | H(N_1(X))-H(N_2(X)) | + | H(XN_1(X))-H(XN_2(X)) |$
$\le ||N_1(X)-N_2(X)||_{tr} \log d_{out} + ||XN_1(X)-XN_2(X)||_{tr} \log d_{in}d_{out}$
$+ 2\,\eta(||N_1-N_2||)$ by Fannes inequality $\quad (\eta(t)=-t \log t)$
$\le 3\,||N_1-N_2|| \log d + 2\,\eta(||N_1-N_2||)$ where $d = \max(d_{in,out})$

## Continuity of C(N) for quantum channels:

f(.,N)

cannot bound $f(.,N_1)-f(.,N_2)$

$$C(N) = \lim_{n\to\infty} \max_{p_x,\rho_x} \frac{1}{n}[S(X)+S(B_1\cdots B_n)-S(XB_1\cdots B_n)]$$

evaluated on $\sum_x p_x|x\rangle\langle x| \otimes N^{\otimes n}(\rho_x)$

Mimic continuity argument for classical channels:

(1) Use the diamond norm (cb trace norm):

$||N_1-N_2||_\diamond := \max_\rho || I\otimes N_1(\rho) - I\otimes N_2(\rho) ||_{tr}$

(2a) $\sum_x p_x|x\rangle\langle x| \otimes N_1^{\otimes n}(\rho_x)$ and $\sum_x p_x|x\rangle\langle x| \otimes N_2^{\otimes n}(\rho_x)$ can be "$n||N_1-N_2||_\diamond$" apart.

(2b) evaluating $S(B_1\cdots B_n)$ on the two states above

Fannes ineq bounds the difference as

"log dim" * distance of the two states + $\eta()$

$\log d_{out}^n = n \log d$ $\quad n||N_1-N_2||$

Solution: tighter bound on entropy difference between two n-use output states.

Main lemma [continuity of output entropy]:

Let $N,M: A\to B$ be quantum channels, $d=\dim(B)$.

R reference system. If $||N-M||_\diamond \le \varepsilon$, $\forall\ \rho_{RA^{\otimes n}}$,

$| S(I\otimes N^{\otimes n}(\rho)) - S(I\otimes M^{\otimes n}(\rho)) | \le n\,[4\,\varepsilon \log d + 2H(\varepsilon)]$

same state — only 1 factor of n

The proof only requires:

- the telescopic sum,
- the triangular inequality, and
- * the Fannes-Alicki inequality (quant-ph/0312081)

$|S(K|L)_\rho - S(K|L)_\sigma| \le \log[\dim(K)]\ |\rho-\sigma|_{tr} + \ldots$

no L

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Proof: Let $\sigma_k = I \otimes N^{\otimes k} \otimes M^{\otimes n-k}(\rho)$

If $|S(\sigma_k)-S(\sigma_{k-1})| \le 4\,\varepsilon \log d + 2\,H(\varepsilon)$ ← now prove this

then $| S(I\otimes N^{\otimes n}(\rho)) - S(I\otimes M^{\otimes n}(\rho)) |$

$= |S(\sigma_n)-S(\sigma_0)|$

$= |\sum_{k=1}^n S(\sigma_k)-S(\sigma_{k-1})|$ telescopic sum

$= \sum_{k=1}^n |S(\sigma_k)-S(\sigma_{k-1})|$ triangular ineq

$\le n\,[4\,\varepsilon \log d + 2H(\varepsilon)]$

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Proof: Let $\sigma_k = I \otimes N^{\otimes k} \otimes M^{\otimes n-k}(\rho)$

$|S(\sigma_k)-S(\sigma_{k-1})|$

$= |S(CB_1 \ldots B_n)_{\sigma k} - S(CB_1 \ldots B_n)_{\sigma k-1}|$

inserting 0

$= |S(CB_1 \ldots B_n)_{\sigma k} - S(CB_1 .. B_{k-1}B_{k+1} .. B_n)_{\sigma k}$
$+ S(CB_1 .. B_{k-1}B_{k+1} .. B_n)_{\sigma k} - S(CB_1 \ldots B_n)|_{\sigma k-1}$

$= |S(CB_1 \ldots B_n)_{\sigma k} - S(CB_1 .. B_{k-1}B_{k+1} .. B_n)_{\sigma k}$
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$\sigma_k$ & $\sigma_{k-1}$ differ only in $B_k$

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$= |S(CB_1 \ldots B_n)_{\sigma k} - S(CB_1 \ldots B_n)_{\sigma k-1}|$

$= |S(CB_1 \ldots B_n)_{\sigma k} - S(CB_1 .. B_{k-1}B_{k+1} .. B_n)_{\sigma k}$
$+ S(CB_1 .. B_{k-1}B_{k+1} .. B_n)_{\sigma k-1} - S(CB_1 \ldots B_n)_{\sigma k-1}|$

$= |S(B_k|CB_1 .. B_{k-1}B_{k+1} .. B_n)_{\sigma k} - S(B_k|CB_1 .. B_{k-1}B_{k+1} .. B_n)_{\sigma k-1}|$

$\le 4\ ||\sigma_k - \sigma_{k-1}||_{tr} \log d + \ldots$ thanks to Alicki-Fannes!

$\le 4\ ||N-M||_\diamond \log d + \ldots$ ← independent of dim of system being conditioned on!!

$\le 4\,\varepsilon \log d + 2\,H(\varepsilon)$

Main lemma [continuity of output entropy]:

Let $N,M: A\to B$ be quantum channels, $d=\dim(B)$.

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$| S(I\otimes N^{\otimes n}(\rho)) - S(I\otimes M^{\otimes n}(\rho)) | \le n\,[4\,\varepsilon \log d + 2H(\varepsilon)]$

Plug in the following:

f(.,N)

$$C(N) = \lim_{n\to\infty} \max_{p_x,\rho_x} \frac{1}{n}[S(X)+S(B_1\cdots B_n)-S(XB_1\cdots B_n)]$$

evaluated on $\sum_x p_x|x\rangle\langle x| \otimes N^{\otimes n}(\rho_x)$

$| C(N_1) - C(N_2) | \le \max_X | f(X,N_1) - f(X,N_2)|$

Get corollary 1: If $|| N_1 - N_2 ||_\diamond \le \varepsilon$, then
$|C(N_1)-C(N_2)| \le 8\,\varepsilon \log d + 4\,H(\varepsilon)$.

**Buy 1 get 2 free:**

- Quantum capacity (Lloyd-Shor-Devetak)

$Q(N) = \lim_{n\to\infty} \max_{\psi} 1/n\ I^{coh}(R>B_1B_2 \cdots B_n)$

evaluated on $I \otimes N^{\otimes n}(\psi_{RA_1A_2 \cdots A_n})$

Corollary 2: If $|| N_1 - N_2 ||_\diamond \leq \varepsilon$, then
$|Q(N_1)-Q(N_2)| \leq 8\ \varepsilon \log d + 4\ H(\varepsilon)$.

- Private classical capacity (Smith-Smolin-Winter)

$C_p(N) = \lim_{n\to\infty} \max_{p_x,\rho_x} 1/n\ [I(X:B_1B_2 \cdots B_n)-I(X:E_1E_2 \cdots E_n)]$

eval on $\sum_x p_x|x\rangle\langle x| \otimes U^{\otimes n}(\rho_{x\ RA_1A_2 \cdots A_n})$

Corollary 3: If $|| N_1 - N_2 ||_\diamond \leq \varepsilon$, then
$|C_p(N_1)-C_p(N_2)| \leq 16\ \varepsilon \log d + 8\ H(\varepsilon)$.

Now what about $Q_2$ [quantum capacity assisted by free 2-way classical communication]?

There's no capacity expression, though $Q_2(N) = E(N)$ (entanglement capacity of the channel)

Guifre Vidal proved distillable entanglement is continuous. That doesn't imply anything for the entanglement capacity of a channel itself.

The result is not applicable, but the idea is.

If two distillable states $\rho_1$, $\rho_2$ are similar, n copies of $\rho_1$ can be converted into $\approx$ n copies $\rho_2$ with LOCC. So, distillable entanglement of $\rho_1$ cannot be much less than that of $\rho_2$. Same with $\rho_1$ and $\rho_2$ interchanged.

Such conversion works for channels too!

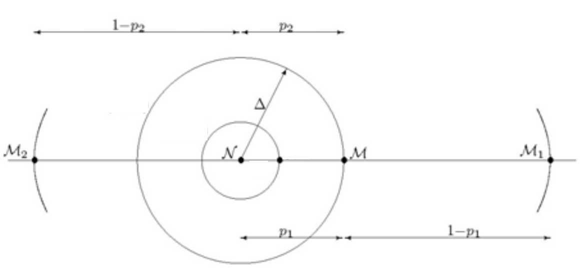
Continuity of $Q_2$ in the interior of $\{Q_2(N)>0\}$

Given channels M, N with $Q_2>0$, $\exists\ M_1, M_2$ such that:

$M = p_1 M_1 + (1-p_1) N$

$N = p_2 M_2 + (1-p_2) M$

$d=\min(d_{in},d_{out})$

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<u>I. Simulate M using N:</u>

(1) Simulate M using $M_1$, N, & free CC

*Receiver tosses n coins with bias $p_1$, tells sender with free CC, the ith coin toss decides whether $M_1$ or N is used for the ith simulation of M*

$np_1\ M_1 + n(1-p_1)\ N \geq n\ M$

(2) Simulate $M_1$ using I using N

$np_1\ (\log d/Q_2(N))\ N \geq np_1\ I \geq np_1\ M_1$

Compose (1) & (2): $n\ [p_1 \log d/Q_2(N)+(1-p_1)]\ N \geq n\ M$

$\therefore\ [p_1 \log d/Q_2(N)+(1-p_1)]\ Q_2(N) \geq Q_2(M)$

Rearranging: $p_1\ (\log d-Q_2(N)) \geq Q_2(M)-Q_2(N)$

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<u>I. Simulate M using N:</u>

$p_1\ (\log d-Q_2(N)) \geq Q_2(M)-Q_2(N)$

<u>II. Simulate N using M:</u>

Applying the same argument to the blue equation:

$p_2\ (\log d-Q_2(M)) \geq Q_2(N)-Q_2(M)$

Thus, $|Q_2(N)-Q_2(M)| \leq \max(p_1,p_2) * \log d$

Continuity of $Q_2$ in the interior of $\{Q_2(N)>0\}$

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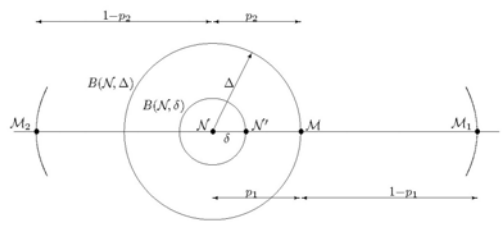
$M = p_1 M_1 + (1-p_1) N$

$N = p_2 M_2 + (1-p_2) M$

$d=\min(d_{in},d_{out})$

$|Q_2(N)-Q_2(M)| \leq \max(p_1,p_2) * \log d$




Replace M by N', then $p_1, p_2 \to 0$ & $|Q_2(N') - Q_2(N)| \to 0$

Same argument holds for $Q_B(N)$ (assisted by free back classical communication).

$Q_B$ of the erasure channel is continuous in the erasure probability p for all p.

So, is continuity "obvious" ?

1. There are pairs of channels $(N_1^n, N_2^n)$ s.t. as $n \to \infty$,
$|| N_1^n - N_2^n ||_\diamond \to 0$, but $|C(N_1^n)-C(N_2^n)|=1$

All the channels take a space spanned by $\{|1\rangle,|2\rangle, ...\}$ to $\{|0\rangle,|1\rangle,|2\rangle, ...\}$

$\forall$ n, $N_1^n = N$, $N(\rho) = tr(\rho)\ |0\rangle\langle 0|$. $C(N) = 0$.

$N_2^n = (1-\frac{1}{\log n})\mathcal{N} + \frac{1}{\log n}\mathrm{id}_n$ . $C(N_2^n) \geq 1$.

↑ identity on $|1\rangle, \cdots, |n\rangle$, acts like N elsewhere

$|| N_1^n - N_2^n ||_\diamond = || N - id_n ||_\diamond / \log n \leq 2 / \log n$

A slightly different type of channels exhibit the same phenomena for Q(N)

So, is continuity "obvious" ?

2. For classical arbitrary varying channels with const input/output dimensions, the capacity (allowing LOCAL randomness) is not continuous when the capacity drops to zero.

3. Unresolved cases:
is $Q_{2 \text{ or } B}(N)$ continuous where $Q_{2 \text{ or } B}(N)=0$?
is $D_{1 \text{ or } 2}(\rho)$ continuous where $D_{1 \text{ or } 2}(\rho)=0$?