The framework of resource inequalities begs the Qn:

Is it possible for $Q(N \otimes I) > Q(N) + Q(I)$?

or for $D(\rho \otimes \underset{\text{ebit}}{\Phi}) > D(\rho) + D(\Phi)$?

ie can noiseless resource help beyond it's stated value?

If so, we have 2 diff quantum capacities

- one defined before
- another that allows borrowing & returning of a catalytic or linear amount of noiseless resources (such as deriving LSD from ♂).

same Qn

We will see $Q,\ Q_{\leftarrow},\ Q_{\leftrightarrow},\ D_{\rightarrow},\ D_{\leftarrow},\ D_{\leftrightarrow}$ are all additive on noiseless resources.

Main idea (in BDSW96) is operational and applies to many capacities, but requires $Q(N) > 0$ to show $Q(N \otimes I_d) = Q(N) + \log d$ etc.

The remaining case: Solved in various informal discussions between Gottesman, Harrow, M. Horodecki, L, Oppenheim, Shor, Winter & never written up.

Thm: If $Q(N) > 0$ then $Q(N \otimes I) = Q(N) + Q(I)$

Pf:

Idea: If $Q(N \otimes I) > Q(N) + Q(I)$, even without "$I$", you can make some using N then use these fabricated I to boost capacity of N — a contradiction.

Detail: Suppose $Q(N \otimes I_d) = Q(N) + \log d + \underset{\text{+ve const}}{\eta}$

$\exists\, n$ & a code C_1 using $(N \otimes I_d)^{\otimes n}$ to produce $n[Q(N \otimes I_d) - \delta_n]$ qbits with error ε_n.

Using N alone, $\exists\, m$ & a code C_2 turning $N^{\otimes m}$ to $m[Q(N) - \delta'_m]$ qbits with error ε'_m

Now consider a code C_3 that uses N alone & does the following:

Run C_2 for some m using $N^{\otimes m}$

and then run C_1 for $n = m\left[\frac{Q(N) - \delta'_m}{\log d}\right]$

using the identity channels produced by C_2 & n uses of N.

Total error: $\delta'_m + \delta_n \quad \rightarrow 0$ as $m \rightarrow \infty$ $(n \rightarrow \infty)$

Total # N consumed: $m + n = m\left[1 + \frac{Q(N) - \delta'_m}{\log d}\right]$

Total # qbit produced: $n\left[Q(N \otimes I_d) - \delta_n\right]$

$$= m\left[\frac{Q(N) - \delta'_m}{\log d}\right]\left[Q(N) + \log d + \eta - \delta_n\right]$$

$$\text{Rate} = \frac{\frac{(Q(N) - \delta'_m)}{\log d}\,(Q(N) + \log d + \eta - \delta_n)}{1 + \frac{Q(N) - \delta'_m}{\log d}}$$

$$= \frac{(Q(N) - \delta'_m)\left(1 + \frac{Q(N) + \eta - \delta_n}{\log d}\right)}{\left(1 + \frac{Q(N) - \delta'_m}{\log d}\right)}$$

$$= \left(Q(N) - \delta'_m\right)\left[1 + \left[\frac{\frac{\eta + \delta'_m - \delta_n}{\log d}}{1 + \frac{Q(N) - \delta'_m}{\log d}}\right]\right]$$

$$\geq \left(Q(N) - \delta'_m\right)\left(1 + \frac{\eta + \delta'_m - \delta_n}{Q(N) + \log d}\right)$$

$> Q(N)$ as $m \rightarrow \infty$, $\delta_n, \delta'_m \rightarrow 0$

Similar argument works for $Q_\leftarrow$, $Q_\leftrightarrow$, $D_\rightarrow$, $D_\leftrightarrow$

when the noisy channel / state has nonzero capacity or distillable entanglement.

Remaining case: it is possible for

- $Q(N) = 0$ but $Q(N \otimes I_d) > \log d$?

$Q_\leftarrow$	:	$Q_\leftarrow$
$Q_\leftrightarrow$	:	$Q_\leftrightarrow$
$D_\rightarrow$		$D_\rightarrow$
$D_\leftrightarrow$		$D_\leftrightarrow$

Ans: no for all.

Thm 1: $D_\rightarrow(\rho \otimes \Phi^{\otimes m}) = D_\rightarrow(\rho) + m$

Pf: $D_\rightarrow(\rho \otimes \Phi^{\otimes m}) = \lim_{n\to\infty} \frac{1}{n} \sup_{\mathcal{D} \in LOCC} I_c(A\rangle B)_{\mathcal{D}((\rho\otimes\Phi^{\otimes m})^{\otimes n})}$

$\forall m, \rho$, $\sup_{\mathcal{D} \in LOCC_\rightarrow} I_c(A\rangle B)_{\mathcal{D}((\rho\otimes\Phi^{\otimes m})^{\otimes n})}$

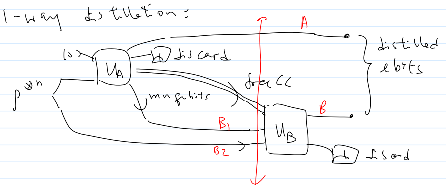
$= \sup_{\substack{\mathcal{D} \in LOCC_\rightarrow \\ + mn \text{ qbits (forward)}}} I_c(A\rangle B)_{\mathcal{D}(\rho^{\otimes n})}$

($\leq$ ED)

($\geq$ TP + free forward CC)

$\leq \sup_{\mathcal{D} \in LOCC_\rightarrow} I_c(A\rangle B)_{\mathcal{D}(\rho^{\otimes n})} + mn$

1-way distillation:



$I_c(A\rangle B)$

$\leq I_c(A\rangle B_2) + mn$

(1 qubit comm from A to B cannot increase $I_c(A\rangle B)$ by more than 1)

$\Big(I_c(A\rangle B) = \underbrace{S(B)}_{\uparrow \text{ by } 1 \text{ at most}} - \underbrace{S(AB)}_{\text{inv}} \Big)$

Cor 1: $Q_\rightarrow(N \otimes I_d) = Q_\rightarrow(N) + Q_\rightarrow(I_d)$

Cor 2: $Q(N \otimes I_d) = Q(N) + Q(I_d)$

Pf (Cor 1): $Q_\rightarrow(N) = \sup_n \frac{1}{n} \max_{|\psi\rangle_{A'^n}} D_\rightarrow\left(I_{A'} \otimes N^{\otimes n} (|\psi\rangle_{A'B^n}) \right)$

$$Q_\rightarrow(N \otimes I_d) = \sup_n \frac{1}{n} \underbrace{\max_{|\tilde\psi\rangle} D_\rightarrow\left(I_{A''} \otimes N^{\otimes n} \otimes I_d^{\otimes n} (|\tilde\psi\rangle) \right)}_{= \max_{|\psi\rangle} D_\rightarrow\left(I_{A'} \otimes N^{\otimes n}(|\psi\rangle) \otimes \Phi^{\otimes n \log d}\right)}$$

Reason:

Given $I_d^{\otimes n}$, one can create $\Phi^{\otimes n \log d}$.

On the other hand, given $\Phi^{\otimes n \log d}$, one can create $I_d^{\otimes n}$ using free forward CC & teleportation (& take $A'' = A \otimes \mathbb{C}^{d^{\otimes n}}$, $|\tilde\psi\rangle = |\psi\rangle$)

$\therefore Q_\rightarrow(N \otimes I_d) = \sup_n \frac{1}{n} \max_{|\psi\rangle} D_\rightarrow\left(I_{A'} \otimes N^{\otimes n}(|\psi\rangle) \otimes \Phi^{\otimes n \log d} \right)$

$\text{(by Thm 1)} = \left[\sup_n \frac{1}{n} \max_{|\psi\rangle} D_\rightarrow\left(I_{A'} \otimes N^{\otimes n}(|\psi\rangle) \right) \right] + \log d$

$= Q_\rightarrow(N) + Q_\rightarrow(I_d)$

Pf (Cor 2): $\forall N$, $Q_\rightarrow(N) = Q(N)$.

$\therefore$ Cor 2 follows from Cor 1.

Thm 2. $\forall \rho$ $D_\leftrightarrow(\rho \otimes \Phi^{\otimes m}) = D_\leftrightarrow(\rho) + D_\leftrightarrow(\Phi^{\otimes m})$

Pf: Suffices to consider $D_\leftrightarrow(\rho) = 0$.

Will prove by contradiction

ie assume $D_\leftrightarrow(\rho \otimes \Phi^{\otimes m}) > m$ and deduce $D_\leftrightarrow(\rho) > 0$.

Suppose $D_\leftrightarrow(\rho \otimes \Phi^{\otimes m}) = m(1+\delta)$ for some const $\delta > 0$

Then $\forall \varepsilon > 0$ $\exists n$ & a protocol P_n that uses n copies of $\rho \otimes \Phi^{\otimes m}$ & 2-way CC to generate $\sigma \approx \Phi^{\otimes m(1+\delta) \cdot n}$ (up to trace distance ε)

depends on ε

Abusing notation, we write $P_n(\rho^{\otimes n} \otimes \Phi^{\otimes mn}) = \sigma$ (ie we denote the TCP map induced by P_n by the same symbol).

Now we provide a protocol to distill ebits from ρ above with nonzero rate (which gives the desired contradiction).

First consider a protocol to check entanglement using random hashing:

Let the 4 bell states on 2 qubit be labeled by 00, 01, 10, 11 where 00 corresponds to Φ.

Suppose Alice & Bob each has k qubits. The 4^k Bell states form a basis for their joint Hilbert space. Represent each basis state as a $2k$-bit string.

In BDSW96, the following can be done by 2 way CC:

① If Alice & Bob share σ_{AB}, they can turn it to a mixture of Bell states, so their state is a classical distribution of "$2k$-bit" strings.

② Learn the parity of any subset of the $2k$ bits by measuring 1 qubit on each side.

So if they measure s qubits each to find the parity of s random subsets

$$\text{prob}(\text{all parity even} \mid \underbrace{\mathrm{tr}(\sigma_{AB}\,\Phi^{\otimes n})=0}_{2k \text{ bit string} \neq 00\cdots00}) \leq \frac{1}{2^s}$$

They output "pass" if all parities are even, else they output "fail".

Let $\tilde{F}_{k,s}$ denote the above map taking $2k$ qubits to $2(k-s)$ qubits.

$$\tilde{F}_{k,s}(\Phi^{\otimes k}) = \Phi^{\otimes (n-k)} \otimes \text{"pass"}$$

$\forall \rho$ s.t. $\mathrm{tr}(\Phi^{\otimes k}\sigma) = 0$,

$$\tilde{F}_{k,s}(\sigma) = \frac{1}{2^s}\tilde{\sigma} \otimes \text{"pass"} + \left(1-\frac{1}{2^s}\right)\tilde{\tilde{\sigma}} \otimes \text{"fail"}$$

where $\tilde{\sigma}, \tilde{\tilde{\sigma}}$ are $2(k-s)$ qubit states.

<u>The distillation protocol for ρ:</u>

Let m, n be as defined before.

$$\text{Let } \Xi = \left[\frac{|00\rangle\langle 00| + |11\rangle\langle 11|}{2}\right]^{\otimes mn} = \frac{\Phi^{\otimes mn}}{2^{mn}} + \left(1-\frac{1}{2^{mn}}\right)\Gamma$$

where $\mathrm{tr}(\Gamma\,\Phi^{\otimes mn}) = 0$, $\Gamma \geq 0$, $\mathrm{tr}\Gamma = 1$.

① Alice & Bob apply P_n to $\rho^{\otimes n}$ & Ξ:

$$\text{Output} = P_n(\rho^{\otimes n} \otimes \Xi)$$
$$= \frac{1}{2^{mn}} P_n(\rho^{\otimes n} \otimes \Phi^{\otimes mn}) + \left(1-\frac{1}{2^{mn}}\right) P_n(\rho^{\otimes n} \otimes \Gamma)$$
$$= \frac{1}{2^{mn}}\sigma + \left(1-\frac{1}{2^{mn}}\right) P_n(\rho^{\otimes n} \otimes \Gamma)$$

$\because$ σ ε-close to $\Phi^{\otimes mn(1+\delta)}$

Output $\frac{\varepsilon}{2^{mn}}$-close to $\frac{1}{2^{mn}}\Phi^{\otimes mn(1+\delta)} + \left(1-\frac{1}{2^{mn}}\right)P_n(\rho^{\otimes n}\otimes\Gamma)$

② Alice & Bob apply $\tilde{F}_{k,s}$ for $k = mn(1+\delta)$.

Output $\frac{\varepsilon}{2^{mn}}$-close to

$$\left[\frac{1}{2^{mn}}\Phi^{\otimes mn(1+\delta)-s} + \left(1-\frac{1}{2^{mn}}\right)\frac{1}{2^s}\widetilde{P_n(\rho^{\otimes n}\otimes\Gamma)}\right] \otimes \text{"pass"}$$
$$+ \left(1-\frac{1}{2^{mn}}\right)\left(1-\frac{1}{2^s}\right)\widetilde{\widetilde{P_n(\rho^{\otimes n}\otimes\Gamma)}} \otimes \text{"fail"}$$

③ Repeat ①&② N times for $N \gg 2^{mn}$.

④ Postselect on states flagged by "pass".

This gives $\approx N\left(\frac{1}{2^{mn}} + \frac{1}{2^s} - \frac{1}{2^{mn+s}}\right)$ copies of

$$\rho' \doteq \frac{1}{2^{mn}}\Phi^{\otimes mn(1+\delta)-s} + \left(1-\frac{1}{2^{mn}}\right)\frac{1}{2^s}P_n(\rho^{\otimes n}\otimes\Gamma)$$

⑤ It is known that ρ' is distillable if

$$\frac{1}{2^{mn}} > \frac{1}{2^{mn(1+\delta)-s}} + \frac{\varepsilon}{2^{mn}} \quad \text{ie } 1 \geq 2^{s-mn\delta} + \varepsilon$$

(singlet fidelity $> \frac{1}{\dim(A)}$, adjusted by error of ρ'.)

Luckily, m = const $\therefore$ we can choose $s = n(md) \times 2$ to ensure the above ($\&$ $D_\leftrightarrow(\rho') = \eta > 0$)

Now n depends on ε we're done with ε (as long as $D_\leftrightarrow(\rho') > 0$). So n is also a const.

⑥ Alice & Bob distill their $N\left(\frac{1}{2^{mn}} + \frac{1}{2^s} - \frac{1}{2^{mn+s}}\right)$ copies of ρ' to get $\approx \eta N\left(\frac{1}{2^{mn}} + \frac{1}{2^s} - \frac{1}{2^{mn+s}}\right)$ ebits.

They've used nN copies of ρ.

$$\therefore D_\leftrightarrow(\rho) \geq \frac{\eta}{n}\left(\frac{1}{2^{mn}} + \frac{1}{2^s} - \frac{1}{2^{mn+s}}\right) > 0$$

$\blacksquare$

NB Previous version of lecture notes failed to stop n from growing so we didn't get $D_\leftrightarrow(\rho) > 0$. (Recycling etc didn't help either.)

Also the alt proof in 9908070 has a problem:

In 9908070v3 lemma 4, it was proved that there is no $S_1 \in$ LOCC st. $S_1(\Phi^{\otimes n} \otimes \rho^{\otimes k}) = \Phi^{\otimes 3n}$

But we're concerned with the existence of a family of protocols $S_1^{(n)}(\Phi^{\otimes n} \otimes \rho^{\otimes k}) \doteq \Phi^{\otimes n(3-\delta_n)}$ where acc is measured by trace distance ε_n

Then single fraction of output when sub $\Phi^{\otimes n}$ by $\frac{1}{4^n} = \frac{1}{4^n} \pm \varepsilon_n$

and $\varepsilon_n << \frac{1}{4^n}$ to ensure the above $\geq \frac{1}{8^n}$ for the needed contradiction.

Cor: $Q_\leftrightarrow(N \otimes I_d) = Q_\leftrightarrow(N) + Q_\leftrightarrow(I_d)$

Pf: $Q_\leftrightarrow(N) = \sup_n \frac{1}{n} D_\leftrightarrow\left(I_{A'} \otimes N^{\otimes n}\left(|\psi\rangle_{A'B^n}\right)\right)$

$$Q_\leftrightarrow(N \otimes I_d) = \sup_n \frac{1}{n} D_\leftrightarrow\left(\underbrace{I_{A'} \otimes N^{\otimes n} \otimes I_d^{\otimes n}(|\tilde{\psi}\rangle)}_{\substack{\text{WLOG} \\ \| \\ I_{A'} \otimes N^{\otimes n}(|\psi\rangle) \otimes \Phi^{\otimes n \log d}}}\right)$$

Since the ebits can be converted to I_d by TP & be used in any other way

$= Q_\leftrightarrow(N) + Q_\leftrightarrow(I_d)$ by Thm on $D_\leftrightarrow$

Cor. $Q_\leftarrow(N \otimes I_d) = Q_\leftarrow(N) + \log d$.

Pf: Suffices to consider $Q_\leftarrow(N) = 0$

But $Q_\leftarrow(N) = 0 \Rightarrow Q_\leftrightarrow(N) = 0$

because if $Q_\leftrightarrow(N) \neq 0$, $D_\leftrightarrow(N) \neq 0$ then use N to general forward classical communication ($C(N) \neq 0$ if $Q_\leftrightarrow(N) \neq 0$) to teleport q. data (this assumes comm in 2-way distillation is at most linear and I think this is true).