Lec 14, Jun 21, 2010

Note Title 21/06/2010

Capacity of transmitting quantum / classical data, using $N$ (many times) as the only nonlocal resource:

LSD: $Q(N) = \sup_n \frac{1}{n} \max_{|\psi\rangle_{RA^{\otimes n}}} I_c(R\rangle B^{\otimes n})_{I\otimes N^{\otimes n}(|\psi\rangle\langle\psi|)}$

HSW: $C(N) = \sup_n \frac{1}{n} \max_{\underbrace{\sum_x p_x |x\rangle\langle x|_R \otimes |\psi_x\rangle\langle\psi_x|_{A^{\otimes n}}}_{\rho_{in}}} S(R:B^{\otimes n})_{I\otimes N^{\otimes n}(\rho_{in})}$

- Consider "assisted" capacities instead, when some extra unlimited nonlocal resource is available for free.

eg $C_E(N)$, $Q_E(N)$, $Q_{\rightarrow}(N)$, $Q_{\leftarrow}(N)$, $Q_{\leftrightarrow}(N)$, $C_{\leftarrow}(N)$

free entanglement; free forward; backward; two-way classical communication

- Assisting resources shouldn't trivialize the task:

eg. entanglement is static – it's independent of the message

eg. classical communication by itself has no quantum capacity

eg. backward comm by itself cannot forward data.

- Motivations
  - upper bounds for unassisted capacities
  - operationally interesting (eg $Q_2(N)$)
  - pieces of a bigger simpler theory
  - inspires unexpected results (like superactivation of $Q$)

LSD: $Q(N) = \sup_n \frac{1}{n} \max_{|\psi\rangle_{RA^{\otimes n}}} I_c(R\rangle B^{\otimes n})_{I\otimes N^{\otimes n}(|\psi\rangle\langle\psi|)}$

HSW: $C(N) = \sup_n \frac{1}{n} \max_{\underbrace{\sum_x p_x |x\rangle\langle x|_R \otimes |\psi_x\rangle\langle\psi_x|_{A^{\otimes n}}}_{\rho_{in}}} S(R:B^{\otimes n})_{I\otimes N^{\otimes n}(\rho_{in})}$

BSST: $C_E(N) = \sup_n \frac{1}{n} \max_{|\psi\rangle_{RA^{\otimes n}}} S(R:B^{\otimes n})_{I\otimes N^{\otimes n}(|\psi\rangle\langle\psi|)}$

$= \max_{|\psi\rangle_{RA}} S(R:B)_{I\otimes N(|\psi\rangle\langle\psi|)}$

Bennett
Shor
Smolin
Thapliyal
0106052

BSST
Cerf & Adami 9609024
Childs, L, Lo 0506039

Shannon: $\underset{\text{Classical}}{C(N)} = \max_{\sum_x p_x |x\rangle\langle x|\otimes|x\rangle\langle x|} S(R:B)_{I\otimes N(\rho_{in})}$

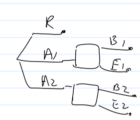
① Additivity:

[Recall lec 11, p18-19, in the proof of additivity of coherent info for degradable channels:
$S(B_1B_2|E_1E_2) \leq S(B_1|E_1) + S(B_2|E_2)$ (due to mono QMI)]

Let $f(N) = \max_{|\psi\rangle_{RA}} S(R:B)_{I\otimes N(|\psi\rangle\langle\psi|)}$

Claim: $f(N_1 \otimes N_2) = f(N_1) + f(N_2)$

Pf: LHS $= \max_{|\psi\rangle_{RA_1A_2}} S(R:B_1B_2)_{I\otimes N_1\otimes N_2(|\psi\rangle\langle\psi|)}$




$= \max_{|\psi\rangle_{RA_1A_2}} S(R) + S(B_1B_2) - S(RB_1B_2)$

$= \max_{|\psi\rangle_{RA_1A_2}} S(B_1B_2E_1E_2) + S(B_1B_2) - S(E_1E_2)$

$\overset{(*)}{=} \max_{|\psi\rangle_{RA_1A_2}} S(B_1B_2|E_1E_2) + S(B_1B_2)$

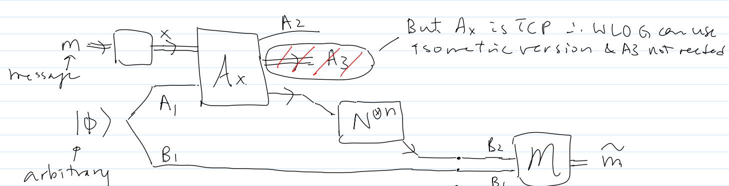
$\leq \max_{|\psi\rangle_{RA_1A_2}} S(B_1|E_1) + S(B_2|E_2) + S(B_1) + S(B_2)$

similar to (*) reversed

$= \max_{|\psi\rangle_{RA_1A_2}} S(\underset{RB_2E_2}{R_1}:B_1) + S(\underset{RB_1E_1}{R_2}:B_2)$

$\leq f(N_1) + f(N_2)$

② General protocol for entanglement assisted classical comm




But $A_x$ is TCP ∴ WLOG can use isometric version & A3 not needed

arbitrary initial entangled state between Alice & Bob.

$\rho_x = ((N^{\otimes n} A_x) \otimes I)(|\psi\rangle\langle\psi|)$

(Without $A_2 A_3 A_1 B_1$, above reduces to an unassisted protocol)

NB. Lack of feedback from Bob to Alice gives the simple structure of the protocol.

③ Converse:

ⓐ Prove sufficiency of considering unitary encoding only (ie no $A_2$)

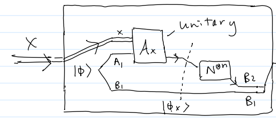
ⓑ Prove converse for unitary encoding

Pf ⓐ For Bob, the most general protocol is same as one in which $A_2$ is sent to him via $\mathcal{R}$ (the completely randomizing map).

By additivity, $C_E(N \otimes \mathcal{R}) = C_E(N) + \cancel{C_E(\mathcal{R})}$

$\therefore$ We can always analyze $N\otimes \mathcal{R}$ instead & focus on unitary encodings

ⓑ The most general protocol with n uses gives a Q-box:




$\rightarrow (I \otimes N^{\otimes n} A_x)(|\phi\rangle\langle\phi|) = \rho_{x\,B_1B_2}$

By the HSW theorem, the classical capacity

$$= \sup_n \frac{1}{n} \max_{A_x, p_x} \chi\{p_x, \rho_x\}\ \forall x,\ (\rho_x)_{B_1} = \rho$$

Facts useful later:

$\forall x,\ \rho_{xB_1} = \mathrm{tr}_{A_1} |\phi\rangle\langle\phi|$, unitarity of $A_x$

$$S(\rho_{xB_1}) = S(\mathrm{tr}_{A_1}|\phi\rangle\langle\phi|) \overset{!}{=} S(\mathrm{tr}_{B_2}|\phi_x\rangle\langle\phi_x|)$$

$$\chi(\{p_x, \rho_{x\,B_1B_2}\}) = S\Big(\sum_x p_x \rho_{xB_1B_2}\Big) - \sum_x p_x S(\rho_{xB_1B_2})$$

$$\overset{\text{sub add}}{\leq} S\Big(\sum_x p_x \rho_{xB_1}\Big) + S\Big(\sum_x p_x \rho_{xB_2}\Big) - \sum_x p_x S(\rho_{xB_1B_2})$$

$$\overset{\text{fact}}{=} \underbrace{\sum_x p_x S(\rho_{xB_1}) - \sum_x p_x S(\rho_{xB_1B_2})} + S\Big(\sum_x p_x \rho_{xB_2}\Big)$$

$$\overset{\text{Lemma to be proved}}{\leq} S\Big(\sum_x p_x \rho_{xB_1}\Big) - S\Big(\underbrace{\sum_x p_x \rho_{xB_1B_2}}\Big) + S\Big(\underbrace{\sum_x p_x \rho_{xB_2}}\Big)$$

③ $\sum_x p_x \rho_{xB_1} = \mathrm{tr}_2 |\phi\rangle\langle\phi|$ but also $\mathrm{tr}_2 \rho$

① this is $I\otimes N^{\otimes}(\rho)$ for $\rho = \sum_x p_x\, I\otimes A_x |\phi\rangle\langle\phi| I \otimes A_x^\dagger$

② this is $N^{\otimes n}(\mathrm{tr}_{B_1}\rho)$

$$\therefore \text{Classical capacity} \leq \sup_n \frac{1}{n} \max_\rho S(\mathrm{tr}_2\rho) + S(N^{\otimes n}(\mathrm{tr}_1\rho)) - S(I\otimes N^{\otimes}(\rho))$$

(replacing max $p_x, \rho_x$)

$$= \sup_n \frac{1}{n} \max_\rho S(R:B^{\otimes n})_{I\otimes N^{\otimes}(\rho)}$$

Finally, by additivity, replace $N^{\otimes n}$ by $N$

Also, can max over pure $\rho$ (a mixed $\rho$ can be purified & including purification is reverse of discarding $\therefore$ QMI nondecreasing).

$$\therefore C_E(N) \leq \max_{|\psi\rangle_{RA}} S(R:B)_{I\otimes N(|\psi\rangle\langle\psi|_{RA})}.$$

④ Direct coding (immediate after proving the "father" next class)

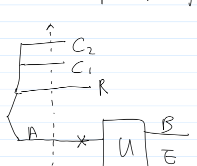
Lemma:

Let $\rho_0, \rho_1$ be density matrices, $0 \leq p_0 \leq 1$, $p_1 = 1 - p_0$, $\rho = p_0\rho_0 + p_1\rho_1$

Then $S(\rho) - S(I\otimes N(\Phi_\rho)) \geq \sum_{i=0}^{1} p_i \left[S(\rho_i) - S(I\otimes N(\Phi_{\rho_i}))\right]$

($\Phi_\rho$ purifies $\rho$; $\Phi_{\rho_i}$ purifies $\rho_i$)

Pf




$\sqrt{p_0}\,|00\rangle_{C_2C_1} |\Phi_{\rho_0}\rangle_{RA}$
$+\sqrt{p_1}\,|11\rangle_{C_2C_1} |\Phi_{\rho_1}\rangle_{RA}$

isometric extension for N

Consider the pure state on $C_2C_1RBE$ (after the channel acts).

LHS ①: $\rho$ = state on A prior to U

$S(\rho) = S(C_2C_1R)$ $\because C_2C_1RA$ pure

LHS ②: $S(I\otimes N(\Phi_\rho)) = S(C_2C_1RB)$

same for all purifications so use $C_1C_2R$ as desired

RHS ①: $S(C_1R) = S(C_1) + \sum_{i=0}^{1} p_i S(\mathrm{tr}_A \Phi_{\rho_i})$

$= S(C_1) + \sum_{i=0}^{1} p_i S(\rho_i)$

$\because R$ purifies $\rho_i$

$\therefore \sum_{i=0}^{1} p_i S(\rho_i) = S(C_1R) - S(C_1)$

RHS ②: $S(C_1RB) = S(C_1) + \sum_{i=0}^{1} p_i S(I\otimes N(\Phi_{\rho_i}))$

$\therefore \sum_{i=0}^{1} p_i S(I\otimes N(\Phi_{\rho_i})) = S(C_1RB) - S(C_1)$

LHS − RHS of claim $= \left[S(C_2C_1R) - S(C_2C_1RB)\right]$

$- \left[(S(C_1R) - S(C_1)) - (S(C_1RB) - S(C_1))\right]$

$= S(C_2C_1R) - S(C_2C_1RB) - S(C_1R) + S(C_1RB)$

$\geq 0$ SSA on $C_2$, $C_1R$, $B$.

egs (for using $C_E(N) = \max_{|\psi\rangle_{RA}} S(R:B)_{I\otimes N(|\psi\rangle\langle\psi|)}$ )

① If $N = I$, $|\psi\rangle_{RA} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is optimal.
We recover superdense coding

② If $N$ = erasure channel w/ error prob $p$:

$I\otimes N(|\psi\rangle\langle\psi|_{RA}) = (1-p)\,|\psi\rangle\langle\psi|_{RB} + p\, tr_B(|\psi\rangle\langle\psi|)\otimes|2\rangle\langle 2|$

$S(R:B) = S(R) + S(B) - S(RB)$

$= S(tr_B(|\psi\rangle\langle\psi|)) + H(p) + (1-p)\, S(tr_R(|\psi\rangle\langle\psi|))$

$- [H(p) + p\, S(tr_B(|\psi\rangle\langle\psi|))]$

$= 2(1-p)\, S(tr_B(|\psi\rangle\langle\psi|))$

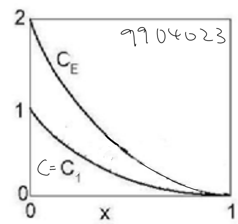
Again, $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ optimal, $C_E(N) = 2(1-p)$.

③ If $N$ is an arbitrary classical channel, $C(N) = C_E(N)$ (save for A3)

From ①-③, tempting to suspect $C_E(N) = C(N)$ for entanglement breaking channels or $C_E(N)/C(N) \le 2 \ \forall N$.

Both refuted by:

④ If $N$ depolarizing channel with prob error $q$,
as $q \to 1$
both $C(N), C_E(N) \to 0$
but $C_E/C \to \infty$ !




Remarks on $C_E$:

① $C_E(N) = 0 \iff N$ trivial ($N(\rho)$ indep of $\rho$)

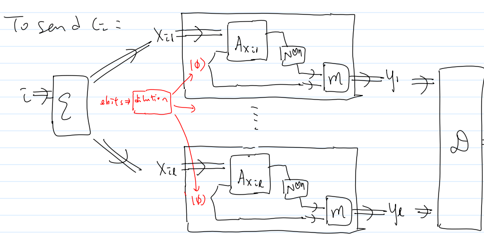
② The most general protocol can use ebits instead of arbitrary entangled state $|\phi\rangle$ by the following argument:
Say, we use a special $|\phi\rangle$ in some $(M,n)$ code with error $\delta_n$.

We can make $l$ copies of $|\phi\rangle$ using $\approx S(tr_1 |\phi\rangle\langle\phi|)$ ebits and $O(\sqrt{l})$ cbits, w/ error $\varepsilon_l' \to 0$ as $l \to \infty$, run the $(M,n)$ code $l$ times in parallel. Each code block is a giant classical channel with prob of error $\varepsilon_n$.

The error (due to $\varepsilon_l'$ & $\varepsilon_n$) can be suppressed by using classical error correcting code. When $\varepsilon_n \to 0$, $n, l$ are large, rate of classical code $\to 1$.

Say, code words of classical code: $c_1 = x_{11}x_{12}\cdots x_{1l}$
$\vdots$ $c_2 = x_{21}x_{22}\cdots x_{2l}$

To send $c_i$:




$[Pr(i \ne j) \to 0]$

If $l \sim \sqrt{n}$, extra cbits needed for dilution can be produced by const $\times \sqrt{n}$ channel uses

$\therefore$ Approx $lM$ messages send via $ln$ + const $\sqrt{n}$ channel uses, As $n \to \infty$, rate $\approx \frac{\log M}{n}$
same as original code.

This method is called "double blocking" (& error reduction is needed in the end).

③ $2Q_E = C_E$

Pf ($\ge$): if $n(C_E - \delta_n)$ cbits can be communicated with error $\varepsilon_n \to 0$ as $n \to \infty$, use these cbits to teleport $\frac{n}{2}(C_E - \delta_n)$ qubits (also with prob error $\le \varepsilon_n$)
$\therefore$ $Q_E \ge \frac{C_E}{2}$.

Pf ($\le$): if $n(Q_E - \delta_n)$ qbits comm w/ error $\varepsilon_n \to 0$ use these qbits to (super)dense-code $2n(Q_E - \delta_n)$ cbits
$\therefore$ $C_E \ge 2Q_E$

④ Reverse Shannon theorem:

ⓐ Consider classical channels.

Think of direct coding as simulating $n\,C(N)$ uses of the identity channel $I_2$ by using $N$ $n$ times.

Turns out simulating $n$ uses of $N$ takes about $n\,C(N)$ uses of identity channel — the "reverse Shannon theorem" (assuming shared randomness free).

Why such a perverted idea? Then $n_1$ uses of $N_1$ simulates $n_2$ uses of $N_2$ for $\frac{n_1}{n_2} \approx \frac{C(N_2)}{C(N_1)}$ — $C(N)$ the only relevant parameter.

ⓑ quantum channels

conceptually, only 1 parameter to describe a channel in the asymptotic setting w/ free ent

· if ebits are free, then $n$ uses of $N$ simulates $n C_E$ cbits.

· turns out if ARBITRARY entangled states are free (not just ebits) then $n C_E$ cbits also simulates $n$ uses of $N$ — the "quantum reverse Shannon Thm" 0912.5537 Bennett Devetak Harrow Shor Winter (starting 2000)

Alt proof: 0912.3805 Berta, Christandl, Renner (2 guest lectures by Will Matthews in July).

Now consider $Q_{\leftrightarrow}(N)$ (q. capacity assisted by 2-way free classical communication (cc)).

Claim $Q_{\leftrightarrow}(N) = E_{\leftrightarrow}(N)$ (ent capacity of $N$ given 2-way cc).

Pf idea "$\geq$": If Alice & Bob can create $n E_{\leftrightarrow}(N)$ ebits w/ error $\leq \varepsilon_n$ by using $N$ $n$ times & 2-way cc

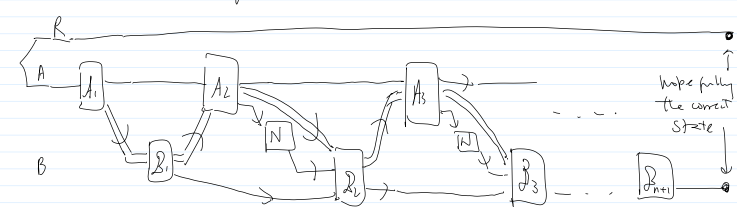
in trace distance

Then Alice can teleport $n E_{\leftrightarrow}(N)$ ebits with $2n E_{\leftrightarrow}(N)$ extra cbits w/ same error.

Pf idea "$\leq$": If $n Q_{\leftrightarrow}(N)$ qubits can be comm from Alice to Bob w/ error (including references) $\leq \varepsilon_n$

she can transmit halves of $n Q_{\leftrightarrow}(N)$ ebits w/ same error bound.

General protocol for $Q_{\leftrightarrow}$:




General protocol for $E_{\leftrightarrow}$:

Same as above but Alice also holds R & only max ent state is seeked at the output.

eg. If $N$ = erasure channel w/ prob error $p$ then $Q_{\leftrightarrow}(N) = 1-p$

$$\left(\begin{array}{l} \text{cf } Q(N) = 1-2p \\ C(N) = 1-p \\ C_E(N) = 2(1-p) \end{array}\right)$$

How?

① Alice sends $n$ halves of $n$ ebits using $N^{\otimes n}$.

② Bob tells Alice which of the $n$ transmissions are erased.

③ They discard the erased ebits

④ They use the ebits + forward cc to teleport $\approx n(1-p)$ qubits.

Unfortunately, neither $Q_{\leftrightarrow}$ nor $E_{\leftrightarrow}$ are known for most channels.

If we aren't impressed with our regularized capacity expressions $Q(N)$ or $C(N)$ we don't even have any expression for $Q_{\leftrightarrow}(N)$.

What about $Q_{\leftarrow}(N)$?

Worse ... no expression & no link to $E_{\leftarrow}$

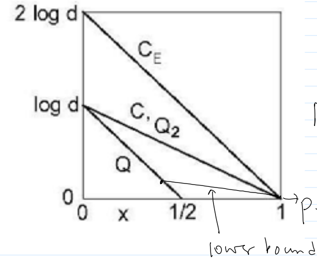
eg. $N$ = erasure channel w/ prob erasure $p$.

We can lower bound $Q_{\leftarrow}(N)$ by $\frac{1-p}{3}$ by protocol:
Same as that for $Q_{\leftrightarrow}$ but since there's no free forward cc, use $N$ to send classical data for teleportation.

Say $n$ uses $\longrightarrow \approx n(1-p)$ ebits
$2n$ uses $\longrightarrow \approx 2n(1-p)$ cbits
$\Big\} \underset{\text{telep}}{\Longrightarrow} n(1-p)$ qbits

Rate: $\frac{n(1-p)}{3n} = \frac{1-p}{3}$.

Capacities for erasure channel as of 9904023

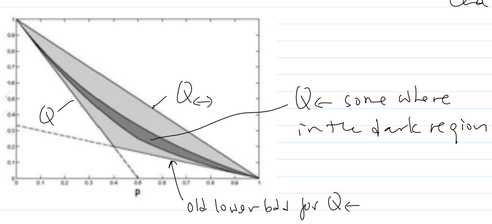



NB. $Q(N) \leq Q_{\leftarrow}(N) \leq Q_{\leftrightarrow}(N)$

$\therefore$ for some channels $Q_{\leftarrow}(N) \gneq Q(N)$

It turns out whether $\exists N$ s.t. $Q_{\leftarrow}(N) \lneq Q_{\leftrightarrow}(N)$ was not easy to determine.

In 0710.5943 (L, Lim, Shor), again for the erasure channel:




- Note that for classical channels, free back comm does not increase the classical capacity.

  How can this be?
  Can prove the SAME converse with feedback
  – see Cover & Thomas (if 1st edition Lec 8.12)

- Classical feedback also doesn't increase $C_E(N)$ for a quantum channel $N$ } ? Winter or BSST

- Quantum feedback (free Q channel from Bob to Alice) increase $C$ to $C_E$, $Q$ to $Q_E$ but no further. (G. Bowen)

What about $Q_{\rightarrow}$?

Claim: $\forall N$ $Q_{\rightarrow}(N) = Q(N)$ (so free forward cc does not increase $Q(N)$)

Pf: