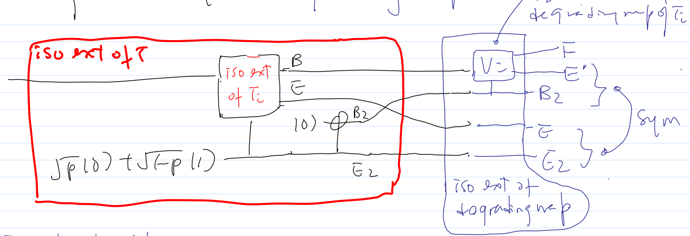
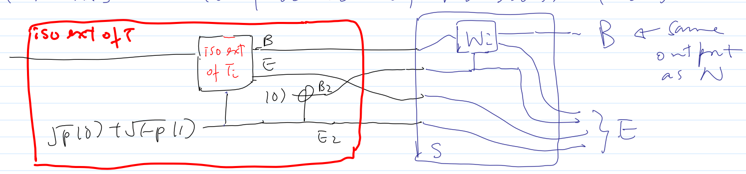
T: degradable by the degrading map:



T extends N: (say $S_i \circ T_i = N_i$, W_i = iso ext of S_i)



Finally, apply Thm 2 to T:

$$Q^{(1)}(T) = Q(T) \le p_1 Q^{(1)}(T_1) + p_2 Q^{(1)}(T_2)$$

By Thm 1, $Q(N) \le Q^{(1)}(T)$

$$\therefore Q(pN_0 + (1-p)N_1) \le p\, Q^{(1)}(T_0) + (1-p)\, Q^{(1)}(T_1)$$

T_i deg ext of N_i

For the depolarizing channel:

① Consider the dephase channel $N(\rho) = (1-p)\rho + pZ\rho Z$

Let $U_1 Z U_1^\dagger = X$, $U_1 X U_1^\dagger = Y$, $U_1 Y U_1^\dagger = Z$

Then the depolarizing channel:

$$N_p = \frac{1}{3} \underset{\text{dephasing w/ Z}}{N} + \frac{1}{3} \underbrace{U_1 \circ N \circ U_1^\dagger}_{\text{dephasing w/ X}} + \frac{1}{3} \underbrace{U_1^2 \circ N \circ U_1^{\dagger 2}}_{\text{dephase w/ Y}}$$

Each of N, $U_1 \circ N \circ U_1^\dagger$, $U_2 \circ N \circ U_2^\dagger$ degradable

$Q^{(1)}$ of each $= 1 - H(p)$

$\therefore Q(N_p) \le 1 - H(p)$

ⓐ recovering Rain's bdd

ⓑ the upper bd for each N_p comes from a deg extension

② Repeat above w/ $N(p)$ = amplitude damping channel

$$Q(N_p) \le H\left(\frac{1-\gamma}{2}\right) - H\left(\frac{\gamma}{2}\right)$$

Where $\gamma = 4\sqrt{1-p}\,(1-\sqrt{1-p})$ (A3)

③ Consider $N_p = (1-4p)I + 4p\, N_{\frac{1}{4}}$ $\left(p \in [0, \frac{1}{4}]\right)$

Both I & $N_{\frac{1}{4}}$ have additive ext

[trivial for I, and use Thm 3 for $N_{\frac{1}{4}}$ & the fact it is known to be anti degradable]

$\therefore$ Thm 4 applies &

$$Q(N_p) \le (1-4p) \cdot 1 + 4p \cdot 0 = 1 - 4p$$

ⓐ recovering Rain's bdd, ⓑ each N_p has deg ext giving the bdd

Putting everything together:

Let $f(p)$ = max convex fcn upper bounded by $1-H(p)$, $1-4p$, $H\left(\frac{1-\gamma}{2}\right) - H\left(\frac{\gamma}{2}\right)$

due to thm 4 & that everything's derived from a deg ext, $Q(N) \le f(p)$