Lec 9, 2010 (part 1)

Note Title 02/06/2010

Last time, we stated ① ② ③ are equivalent:

① $\forall N_1, N_2 \quad \chi^{(1)}(N_1 \otimes N_2) = \chi^{(1)}(N_1) + \chi^{(1)}(N_2)$

② $\forall \rho_1, \rho_2 \quad E_F(\rho_1 \otimes \rho_2) = E_F(\rho_1) + E_F(\rho_2)$

③ $\forall N_1, N_2 \quad S_{min}(N_1 \otimes N_2) = S_{min}(N_1) + S_{min}(N_2)$

Define $\underbrace{\chi_\rho}_{\text{given}}(N) = \max_{\substack{\{p_i, \rho_i\} \\ \text{s.t. } \sum_i p_i \rho_i = \rho}} S(N(\rho)) - \sum_i p_i S(N(\rho_i))$

Let ⓪ $\forall \rho_1 \rho_2 N_1 N_2, \quad \chi_{\rho_1 \otimes \rho_2}(N_1 \otimes N_2) = \chi_{\rho_1}(N_1) + \chi_{\rho_2}(N_2)$

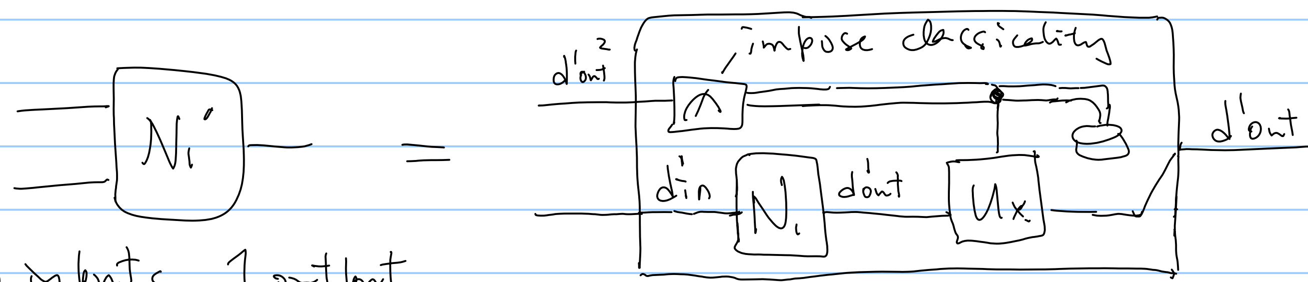
Plan: $\underbrace{① \Rightarrow ③}_{\text{simple}} \underbrace{\Rightarrow ⓪}_{\text{real pain}} \underbrace{\Rightarrow ①}_{\text{missing}}$, $\underbrace{② \Longleftrightarrow ⓪}_{\text{simple}}$

① ⇒ ③ Add of $\chi^{(1)}$ ⇒ Add of S_{min}

Given any N_1 & N_2, want $S_{min}(N_1 \otimes N_2) = S_{min}(N_1) + S_{min}(N_2)$

Idea: construct N_1' out of N_1, N_2' out of N_2

and use $\chi^{(1)}(N_1' \otimes N_2') = \chi^{(1)}(N_1') + \chi^{(1)}(N_2')$



2 inputs, 1 output

$$\text{where } \frac{1}{d_{out}'^{\,2}} \sum_{x=1}^{d_{out}'^{\,2}} U_x \, \eta \, U_x^{\dagger} = \frac{I}{d_{out}'}$$

$$\therefore N_1'(|x\rangle\langle x| \otimes \rho) = U_x N_1(\rho) U_x^{\dagger}$$

Recall optimal ensemble has only pure states.

Because of special structure of N_1'

the ith signal state should look like $|X_i\rangle|\psi_i\rangle$

Also, to max $\chi^{(1)}(N_1')$:

① max $\underbrace{S(\text{average output})}$ ② min ave $\underbrace{S\left(\begin{array}{c}\text{individual}\\\text{outputs}\end{array}\right)}$

This can be maximized by choosing x_i uniformly while taking $|\psi_i\rangle$ the same $\forall i$

ith output: $U_{x_i} N_1(|\psi_i\rangle\langle\psi_i|)\, U_{x_i}^\dagger$

entropy $= S(N_1(|\psi_i\rangle\langle\psi_i|))$

$\therefore$ take $|\psi_i\rangle = |\psi_1\rangle$ $\forall i$

where $\rho = |\psi_1\rangle\langle\psi_1|$ min $S(N_1(\rho))$

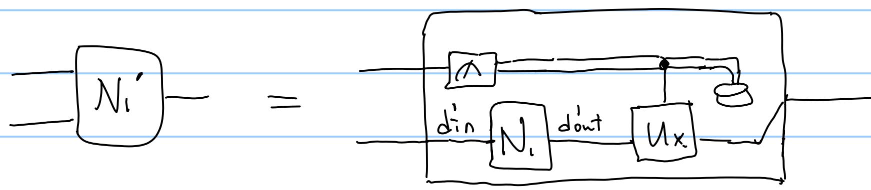
simultaneously optimized

$$\therefore \chi^{(1)}(N_1) = \log\left(d'_{out}\right) - \min_{|\psi\rangle} S\left(N_1\left(|\psi_1\rangle\langle\psi_1|\right)\right)$$

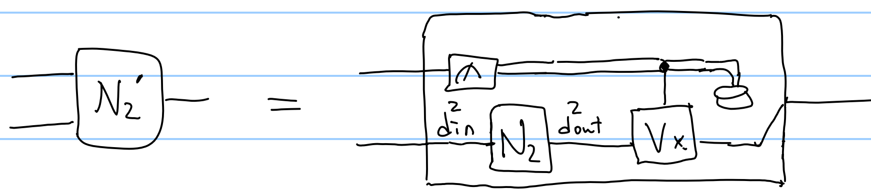
Similarly construct N_2' out of N_2 in the same way.

(Note: N_1, N_2 can have diff input / output dims.)

$$\frac{1}{{d'_{out}}^2} \sum_{x=1}^{{d'_{out}}^2} U_x \eta U_x^\dagger = \frac{I}{d'_{out}}$$



$$N_1'\left(|x\rangle\langle x| \otimes \rho_1\right) = U_x N_1(\rho) U_x^\dagger$$



$$N_2'\left(|x\rangle\langle x| \otimes \rho_2\right) = V_x N_2(\rho) V_x^\dagger$$

$$\frac{1}{{d^2_{out}}^2} \sum_{x=1}^{{d^2_{out}}^2} V_x \eta V_x^\dagger = \frac{I}{d^2_{out}}$$

So, consider $\mathcal{N}_1' \otimes \mathcal{N}_2'$ (which is also $(\mathcal{N}_1 \otimes \mathcal{N}_2)'$):

out the optimal ensemble has $d_{out}^{1\,2} \times d_{out}^{2\,2}$ states

of the form $|x_1\rangle |x_2\rangle |\psi\rangle$, drawn uniformly.

where $x_1 = 1, \cdots, d_{out}^{1\;2}$, $x_2 = 1, \cdots, d_{out}^{2\;2}$

$$\underbrace{\rho = |\psi\rangle\langle\psi|}_{\text{lives in the joint input spaces of } \mathcal{N}_1 \,\&\, \mathcal{N}_2} \min S(\mathcal{N}_1 \otimes \mathcal{N}_2(\rho)).$$

$$\chi^{(1)}(\mathcal{N}_1 \otimes \mathcal{N}_2) = \log d_{out}^1 + \log d_{out}^2 - \min_{|\psi\rangle} S(\mathcal{N}_1 \otimes \mathcal{N}_2(|\psi\rangle\langle\psi|))$$

But $\chi^{(1)}(N_1') = \log d_{out}' - \min_{|\psi_1\rangle} S(N_1'(|\psi_1\rangle\langle\psi_1|))$

$\chi^{(1)}(N_2') = \log d_{out}^2 - \min_{|\psi_2\rangle} S(N_2'(|\psi_2\rangle\langle\psi_2|))$

$\therefore$ If ① is true ie $\chi^{(1)}$ is additive for ANY 2 channels

Then it holds for N_1' & N_2'

$\therefore \chi^{(1)}(N_1' \otimes N_2') = \chi^{(1)}(N_1') + \chi^{(1)}(N_2')$

Cancelling the $\log(d_{out})$ terms & negating gives:

$\min_{|\psi\rangle} S(N_1 \otimes N_2(|\psi\rangle\langle\psi|)) = \min_{|\psi_1\rangle} S(N_1(|\psi_1\rangle\langle\psi_1|)) + \min_{|\psi_2\rangle} S(N_2(|\psi_2\rangle\langle\psi_2|))$

and the above holds for ANY N_1 & N_2 $\therefore$ ③ is true.

Trying 3 $\Rightarrow$ 0 or 3 $\Rightarrow$ 1 :

No luck ....

Showing ② $\Longleftrightarrow$ ⓪ :

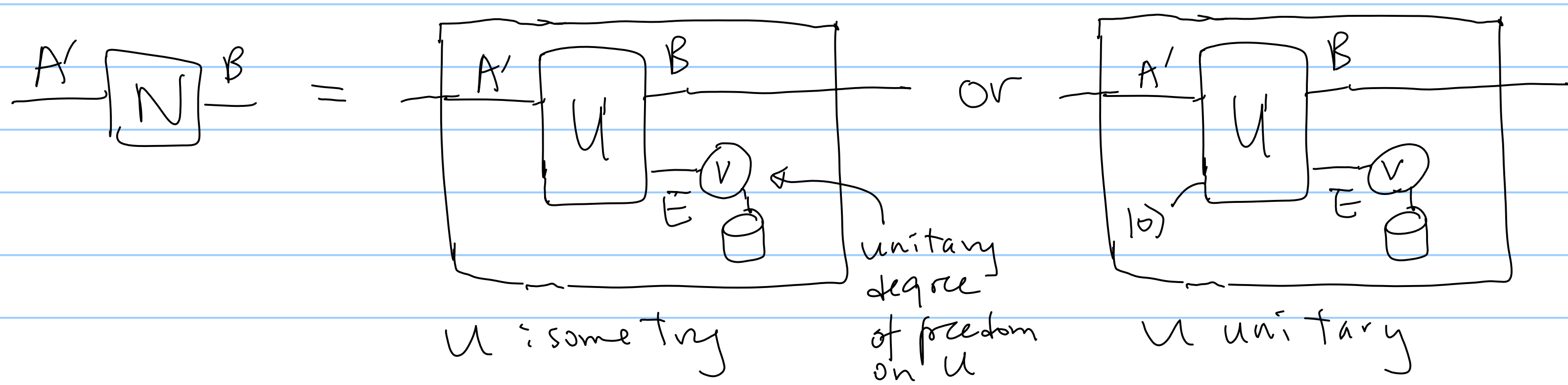
Ingredients:

(i) Stinespring dilation / isometric extension of of $\mathcal{N}$:

Given $\mathcal{N}$ (taking sys A' to B)

$\exists$ $\mathcal{U}$ (taking sys A' to BE)

s.t. $\mathcal{N}(\rho) = \mathrm{tr}_E\, U \rho U^\dagger$



(ii) Matsumoto, Shimono, Winter:

Given ρ and N.

$$\chi_\rho(N) = \max_{\{p_i, \rho_i\}} S(\underbrace{N(\rho)}_{\text{fixed}}) - \sum_i p_i S(N(\rho_i))$$

ρ_i can be chosen pure

$$= S(N(\rho)) - \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i S(N(|\psi_i\rangle\langle\psi_i|))$$

$$= S(N(\rho)) - \min_{\{p_i, |\psi_i\rangle\}} \underbrace{\sum_i p_i E(U|\psi_i\rangle)}$$

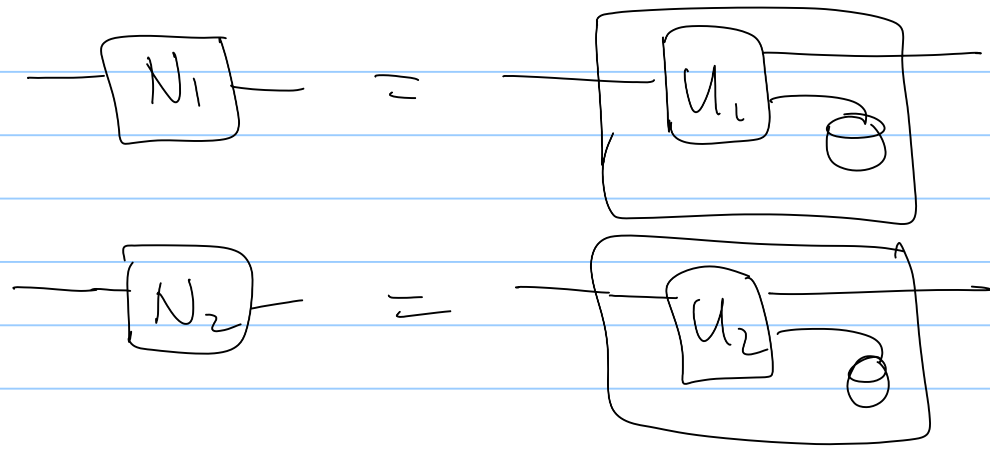
entanglement across BE

E_f of $\sum_i p_i U|\psi_i\rangle\langle\psi_i| U^\dagger$

$= U\rho U^\dagger = \rho - \boxed{U} {}^{B}_{E}$

Now, given N_1, N_2, ρ_1, ρ_2

let



arbitrary on the joint input spaces

$$\chi_{\rho_1 \otimes \rho_2}(N_1 \otimes N_2) = \max_{\substack{\{p_i, \eta_i\} \\ \sum_i p_i \eta_i = \rho_1 \otimes \rho_2}} \Big[S\big(N_1 \otimes N_2(\rho_1 \otimes \rho_2)\big) - \sum_i p_i S\big(N_1 \otimes N_2(\eta_i)\big) \Big]$$

// MSW

$$S\big(N_1 \otimes N_2(\rho_1 \otimes \rho_2)\big) - E_f\big(U_1 \otimes U_2(\rho_1 \otimes \rho_2)\big)$$

$$\parallel$$

$$S(N_1(\rho_1)) + S(N_2(\rho_2)) - E_f(U_1 \rho_1 U_1^\dagger \otimes U_2 \rho_2 U_2^\dagger)$$

$$= \chi_{\rho_1}(N_1) + \chi_{\rho_2}(N_2) + \left[\begin{array}{l} E_f(U_1 \rho_1 U_1^\dagger) + E_f(U_2 \rho_2^\dagger U_2^\dagger) \\ - E_f(U_1 \rho_1 U_1^\dagger \otimes U_2 \rho_2 U_2^\dagger) \end{array} \right]$$

$\therefore$ If ② holds (E_f additivity) then ⓪ holds $\forall N_1 N_2 \rho_1 \rho_2$.

Converse: given σ_1 & σ_2, we can always find ρ_1, ρ_2, U_1, U_2 s.t. $\sigma_1 = U_1 \rho_1 U_1^\dagger$, $\sigma_2 = U_2 \rho_2 U_2^\dagger$

$\therefore$ ② $\Leftrightarrow$ ⓪ But unfortunately, does not give ② $\Leftrightarrow$ ①
since in $\chi^{(1)}$ we cannot control what ρ is.

Claim of ① $\Rightarrow$ ② were made based on finding a channel for which one can control the optimal average output but no luck in my reading (with tight time constraints).