Lec 9, part II

Note Title 02/06/2010

<u>Moving on to transmit quantum data via quantum channels</u>

As in the case to send classical data via a classical / quantum channel channel noise degrades the transmitted state, and error correcting codes are used to suppress the degradation (at lower data rate).

<u>Def</u>: given quantum channel N, we say R is an achievable rate if $\forall n$, $\exists$ $(M = 2^{nR}, n)$ code transmitting an M-dim QUANTUM state with n uses of N, error $\leq \varepsilon_n$ & $\varepsilon_n \longrightarrow 0$ as $n \to \infty$

$$\rho \xrightarrow[M = 2^{nR} \text{ dims}]{} \underbrace{\boxed{\mathcal{E}^{(n)}}}_{\text{encoder}} \longrightarrow \boxed{N^{\otimes n}} \longrightarrow \underbrace{\boxed{\mathcal{D}^{(n)}}}_{\text{decoder}} \longrightarrow \tilde{\rho} \overset{\varepsilon_n}{\approx} \rho$$

$$\boxed{\begin{array}{l} \text{Quantum Capacity} \\ Q(N) \\ \quad \shortparallel \\ \text{supremum overall} \\ \text{achievable rates} \end{array}}$$

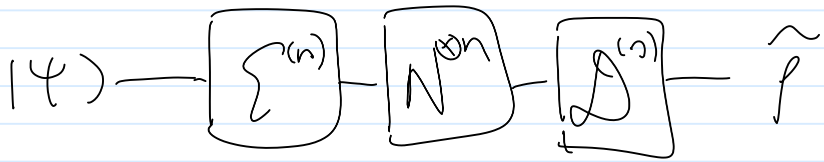
Problem: how to define $\tilde{\rho} \overset{\varepsilon_n}{\approx} \rho$?

There are many diff defs but luckily they give the same capacity.

See eg 0311037 by Kretschmann & Werner.

We can demand any of the following:

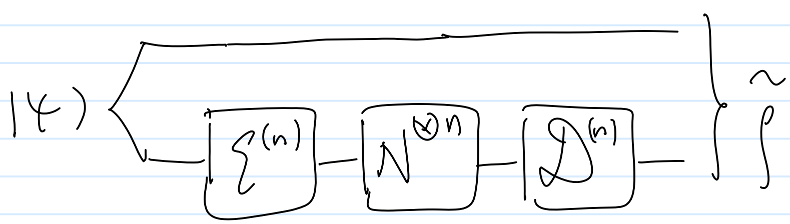
1



(a) $\underset{|\psi\rangle}{\text{Average}} \langle\psi|\tilde{\rho}|\psi\rangle \geq 1-\varepsilon_n$

(b) $\forall\, |\psi\rangle,\ \langle\psi|\tilde{\rho}|\psi\rangle \geq 1-\varepsilon_n$

2



(a) for $|\psi\rangle =$ max ent state $\langle\psi|\tilde{\rho}|\psi\rangle \geq 1-\varepsilon_n$

(b) $\forall\, |\psi\rangle,\ \langle\psi|\tilde{\rho}|\psi\rangle \geq 1-\varepsilon_n$

Can also replace $\langle\psi|\tilde{\rho}|\psi\rangle \geq 1-\varepsilon_n$ by $\| \, |\psi\rangle\langle\psi| - \tilde{\rho} \, \|_{tr} \leq \varepsilon_n$

Def ① revolves about "preserving" the input state

traditionally the easier one to show / think about.

Def ② revolves about $-\boxed{\boxed{\mathcal{E}^{(n)}} - \boxed{\mathcal{N}^{(n)}} - \boxed{\mathcal{D}^{(n)}}}-$ "simulating"

the identity channel on 2^{nR} - dim Hilbert space.

When used w/ the trace distance criteria,

(2a) tells us how good the simulation is, measured by the trace distance of the "Choi-Jamiotkowski" states of the simulating & the simulated channels.

[CJ state of N is just " $|\psi\rangle \begin{cases} \text{———} \\ -\boxed{N}- \end{cases}$ " for $|\psi\rangle$ = max ent state

This completely specifies the channel N.]

(2b) tells use the distance between the simulated & the simulating channel in the "diamond norm."

$$\| N_1 - N_2 \|_\diamond := \max_{|\psi\rangle} \| I \otimes N_1 (|\psi\rangle\langle\psi|) - I \otimes N_2 (|\psi\rangle\langle\psi|) \|_{tr}$$

2 channels having small diamond norm is the most demanding way to say they're similar.

Aside: some researchers use the "cb-norm"

Besides the Stinespring dilation & the CJ state a channel N can also be written in the "kraus form"

$$N(\rho) = \sum_K A_K \rho A_K^\dagger \quad \text{where} \quad \sum_k A_k^\dagger A_k = I$$

$N^*(\eta) = \sum_k A_k^\dagger \eta A_k$ is call the "adjoint" of N.

If N evolves state ρ in the Schroedinger picture then N^* evolves observables η in the Heisenberg picture.

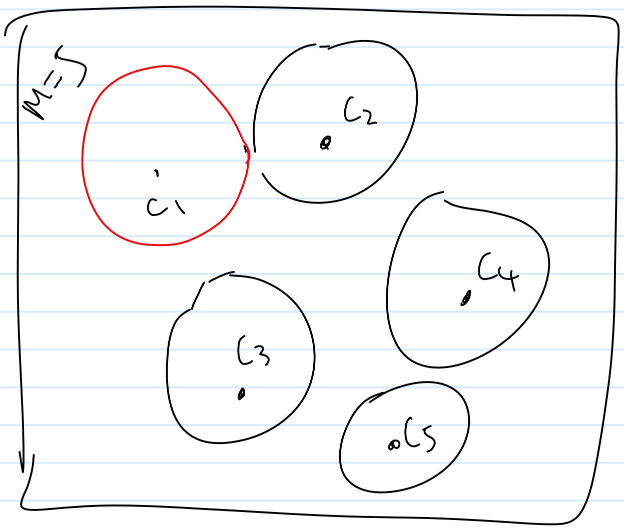
$$\| N_1 - N_2 \|_\diamond = \| N_1^* - N_2^* \|_{cb}$$

Luckily, if $N_1 \approx I$, $N_1^* \approx I$

so the diamond & cb norm are similar when measuring the accuracy of approx the identity.

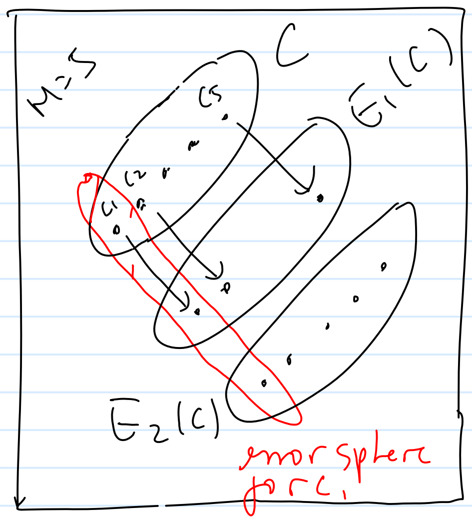
Qn: What is an (M, n) code for quantum data?

In the classical case:



error "sphere" for different codewords do not overlap.

Better view:



Codewords form a subspace C & "likely errors" translate C to orthogonal spaces.

Quantum case

exactly like this

Except you can have different errors E_1 & E_2 taking C to the same space.

When this happens C is called "degenerate".

To be concrete:

9 bit degenerate code:

$|0\rangle_L \sim (|000\rangle + |111\rangle)^{\otimes 3}$

$|1\rangle_L \sim (|000\rangle - |111\rangle)^{\otimes 3}$

Assuming error on 2 or more qubits "unlikely", then the likely errors are linear combination of Pauli operators on up to 1 qubit.

Z_1, Z_2, Z_3 all act the same on C (same for Z_4, Z_5, Z_6, Z_7, Z_8, Z_9).

Z_1, Z_4, Z_7, X_{1-9} Take C to ortho spaces, i.e. They're "distinguishable" thus correctable.

Alt: code C is "+1" eigenspace of commuting operators:

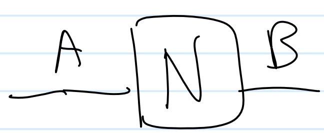
$X_1X_2X_3X_4X_5X_6$, $X_4X_5X_6X_7X_8X_9$, Z_1Z_2, Z_2Z_3, Z_4Z_5, Z_5Z_6, Z_7Z_8, Z_8Z_9

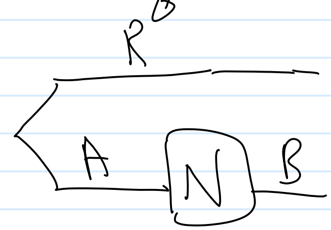
5 bit nondegenerate code:

+1 eigenspace of $X_1 Z_2 Z_3 X_4$, $X_2 Z_3 Z_4 X_5$, $X_3 Z_4 Z_5 X_1$, $X_4 Z_5 Z_1 X_2$

Back to quantum capacity.

Reference system (not achievable rate R)

Let 

and 

called the coherent info
actually $-S(R|B)$

Def: $$Q^{(1)}(N) = \max_{|\psi\rangle} I_c(R>B)_{I\otimes N(|\psi\rangle\langle\psi|)}$$

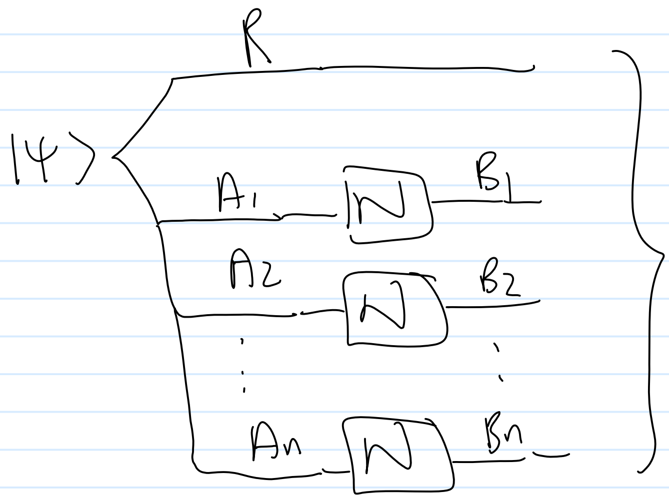
Where $I_c(R>B) = [S(B) - S(RB)]$ evaluated on $I\otimes N(|\psi\rangle\langle\psi|)$

Def: $$Q^{(n)}(N) = \frac{1}{n} Q^{(1)}(N^{\otimes n})$$

The LSD thm:

$$Q(N) = \sup_n Q^{(n)}(N)$$

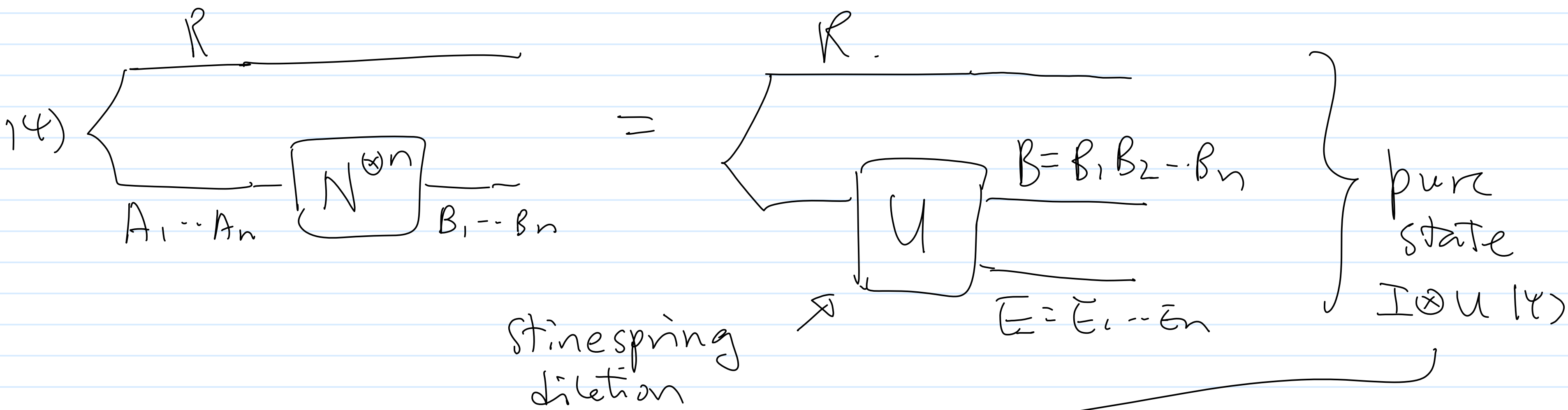
In other words:



$I_R \otimes N^{\otimes n}(|\psi\rangle\langle\psi|)$.

- Calculate $S(B_1 \ldots B_n) - S(R B_1 \ldots B_n)$
- divide by n
- sup over n.

So, what is $I_c(R\rangle B_1\cdots B_n)$?



$$I_c(R\rangle B) = S(B) - S(BR) = S(ER) - S(E)$$

$$= \frac{1}{2}\Big(\big[S(B) - S(BR) + S(R)\big] + \big[S(ER) - S(E) - S(R)\big]\Big)$$

$$= \frac{1}{2}\big[S(B:R) - S(E:R)\big]$$

When transmitting classical data,

we max $S(B:R)$ (evaluated on $\sum_x \underbrace{p_x |x\rangle\langle x|}_{R} \otimes \underbrace{N^{\otimes n}(\rho_x)}_{B}$)

For quantum data

not only we need to max Bob correlation with R,

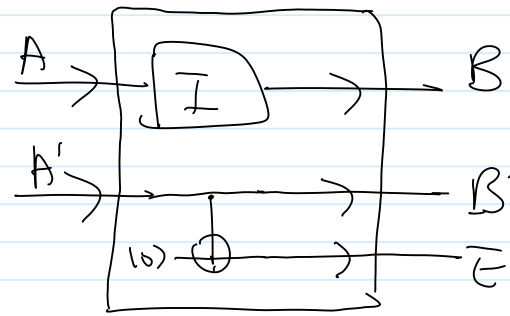
he needs to take out a portion of his data

that's correlated with Eve. The QMI counts all

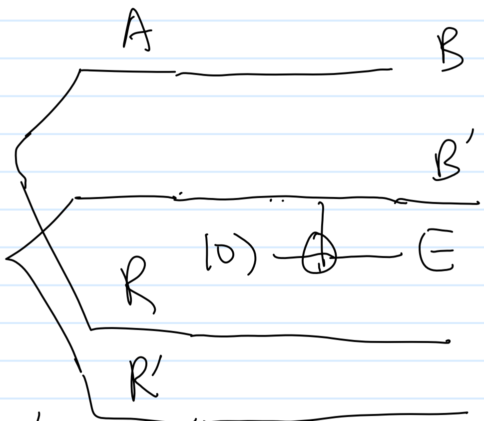
correlation (classical + quantum) and the classical portion

is quantified by $S(R:E)$.

eg. let $N = 1$ qbit + 1 cbit. Stinespring's dilation



Input $= \left(|00\rangle + |11\rangle\right)_{AR} \left(|00\rangle + |11\rangle\right)_{A'R'}$



Output $= \underbrace{\left(|00\rangle + |11\rangle\right)_{RB}}_{S(R:B)=2} \otimes \underbrace{\left(|000\rangle + |111\rangle\right)_{R'B'E}}_{S(R':B')=1}$

$S(R':E)=1$

$S(RR':BB')=3, \quad S(RR':E)=1$

$I_c(RR' \rangle BB') = \frac{1}{2}(3-1) = 1$ (expected).

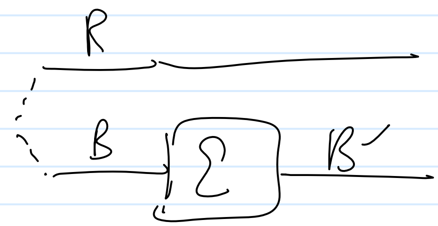
Properties of the coherent information $I(R \rangle B)$:

① Invariant under local unitary on R & B

② Invariant under attaching ancillas in pure states

③ Non increasing under local operation on B's side

Pf by monotonicity of QMI:



$$S(R:B)_\rho \geq S(R:B')_{I\otimes\mathcal{E}(\rho)}$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$S(R) + S(B) - S(RB) \qquad\qquad S(R) + S(B') - S(RB')$$

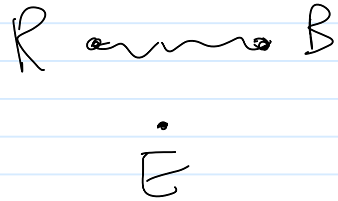
same, since evaluated on $\mathrm{tr}_B\,\rho = \mathrm{tr}_{B'}[I\otimes\mathcal{E}(\rho)]$

$$\therefore\; I(R\rangle B)_\rho \geq I(R\rangle B')_{I\otimes\mathcal{E}(\rho)}.$$

quantum data processing inequality

④ Discarding a system on R may increase / decrease $I(R\rangle B)$.

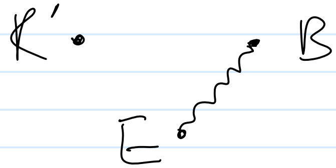
eg1.



Initial state $= (|00\rangle + |11\rangle)_{RB} |0\rangle_E$

$I(R\rangle B) = \underset{1}{S(B)} - \underset{0}{S(RB)} = 1$

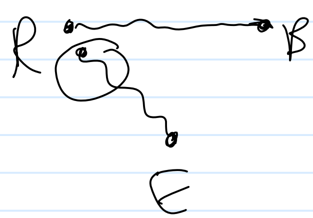
Discarding R (giving it to E)



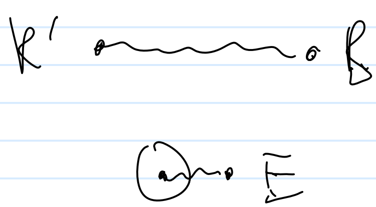
final state $= |0\rangle_R (|00\rangle + |11\rangle)_{BE}$

$I(R'\rangle B) = \underset{1}{S(B)} - \underset{1}{S(R'B)} = 0.$

eg2.



$I(R\rangle B) = S(B) - S(BR) = 1 - 1 = 0$



$I(R\rangle B) = 1 - 0 = 1.$