Lec 9, part II

Note Title 02/06/2010

Moving on to transmit quantum data via quantum channels

As in the case to send classical data via a classical / quantum channel, channel noise degrades the transmitted state, and error correcting codes are used to suppress the degradation (at lower data rates).

Def: given quantum channel N, we say R is an achievable rate if $\forall n$, $\exists$ ($M = 2^{nR}$, n) code transmitting an M-dim QUANTUM state with n uses of N, error $\le \epsilon_n$ & $\epsilon_n \to 0$ as $n \to \infty$

$$\rho \xrightarrow[M=2^{nR}\text{ dims}]{} \underbrace{\boxed{\mathcal{E}^{(n)}}}_{\text{encoder}} \to \boxed{N^{\otimes n}} \to \underbrace{\boxed{\mathcal{D}^{(n)}}}_{\text{decoder}} \to \tilde\rho \overset{\epsilon_n}{\approx} \rho$$

Quantum Capacity $Q(N)$ = supremum over all achievable rates

Problem: how to define $\tilde\rho \overset{\epsilon_n}{\approx} \rho$?

There are many diff defs but luckily they give the same capacity.

See eg 0311037 by Kretschmann & Werner.

We can demand any of the following:

① $|\psi\rangle - \boxed{\mathcal{E}^{(n)}} - \boxed{N^{\otimes n}} - \boxed{\mathcal{D}^{(n)}} - \tilde\rho$

ⓐ Average$_{|\psi\rangle}$ $\langle\psi|\tilde\rho|\psi\rangle \ge 1-\epsilon_n$

ⓑ $\forall |\psi\rangle$, $\langle\psi|\tilde\rho|\psi\rangle \ge 1-\epsilon_n$

② $|\psi\rangle$ with one half through $- \boxed{\mathcal{E}^{(n)}} - \boxed{N^{\otimes n}} - \boxed{\mathcal{D}^{(n)}} -$, output $\tilde\rho$

ⓐ for $|\psi\rangle$ = max ent state $\langle\psi|\tilde\rho|\psi\rangle \ge 1-\epsilon_n$

ⓑ $\forall |\psi\rangle$, $\langle\psi|\tilde\rho|\psi\rangle \ge 1-\epsilon_n$

Can also replace $\langle\psi|\tilde\rho|\psi\rangle \ge 1-\epsilon_n$ by $\| |\psi\rangle\langle\psi| - \tilde\rho \|_{tr} \le \epsilon_n$

Def ① revolves about "preserving" the input state traditionally the easier one to show / think about.

Def ② revolves about $- \boxed{\mathcal{E}^{(n)}} - \boxed{N^{(n)}} - \boxed{\mathcal{D}^{(n)}} -$ "simulating" the identity channel on 2^{nR}-dim Hilbert space.

When used w/ the trace distance criteria,

②ⓐ tells us how good the simulation is, measured by the trace distance of the "Choi-Jamiołkowski" states of the simulating & the simulated channels.

[CJ state of N is just " $|\psi\rangle$ with one half through $\boxed{N}$ " for $|\psi\rangle$ = max ent state

This completely specifies the channel N.]

②ⓑ tells use the distance between the simulated & the simulating channel in the "diamond norm".

$$\| N_1 - N_2 \|_\diamond := \max_{|\psi\rangle} \| I\otimes N_1(|\psi\rangle\langle\psi|) - I\otimes N_2(|\psi\rangle\langle\psi|) \|_{tr}$$

2 channels having small diamond norm is the most demanding way to say they're similar.

Aside: some researchers use the "cb-norm"

Besides the Stinespring dilation & the CJ state a channel N can also be written in the "kraus form"

$N(\rho) = \sum_k A_k \rho A_k^\dagger$ where $\sum_k A_k^\dagger A_k = I$

$N^*(\eta) = \sum_k A_k^\dagger \eta A_k$ is call the "adjoint" of N.

If N evolves state ρ in the Schroedinger picture then N^* evolves observables η in the Heisenberg picture.

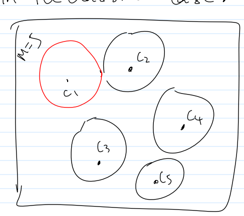
$$\| N_1 - N_2 \|_\diamond = \| N_1^* - N_2^* \|_{cb}$$

Luckily, if $N_1 \approx I$, $N_1^* \approx I$

so the diamond & cb norm are similar when measuring the accuracy of approx the identity

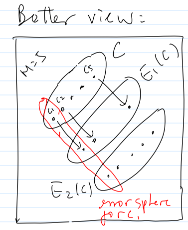
Qn: What is an (M, n) code for quantum data?

In the classical case:



error "sphere" for different codewords do not overlap.

Better view:



Codewords form a subspace C & "likely errors" translate C to orthogonal spaces.

Quantum case

exactly like this

Except you can have different errors E_1 & E_2 taking C to the same space.

When this happens C is called "degenerate".

To be concrete:

9 bit degenerate code:

$|0\rangle_L \sim (|000\rangle + |111\rangle)^{\otimes 3}$

$|1\rangle_L \sim (|000\rangle - |111\rangle)^{\otimes 3}$

Assuming error on 2 or more qubits "unlikely", then the likely errors are linear combination of Pauli operators on up to 1 qubit.

Z_1, Z_2, Z_3 all act the same on C (same for Z_4, Z_5, Z_6, Z_7, Z_8, Z_9).

Z_1, Z_4, Z_7, X_{1-9} Take C to ortho spaces, $\therefore$ they're "distinguishable" thus correctable.

Alt: code C is "+1" eigenspace of commuting operators:

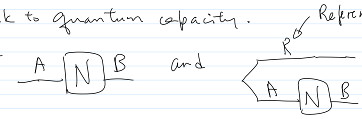
$X_1X_2X_3X_4X_5X_6$, $X_4X_5X_6X_7X_8X_9$, Z_1Z_2, Z_2Z_3, Z_4Z_5, Z_5Z_6, Z_7Z_8, Z_8Z_9

5 bit nondegenerate code:

+1 eigenspace of $X_1Z_2Z_3X_4$, $X_2Z_3Z_4X_5$, $X_3Z_4Z_5X_1$, $X_4Z_5Z_1X_2$

Back to quantum capacity.

Let



called the coherent info actually $-S(R|B)$

Def: $Q^{(1)}(N) = \max_{|\psi\rangle} I_c(R>B)_{I\otimes N(|\psi\rangle\langle\psi|)}$

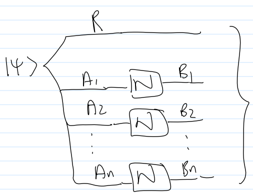
Where $I_c(R>B) = [S(B) - S(RB)]$ evaluated on $I\otimes N(|\psi\rangle\langle\psi|)$

Def: $Q^{(n)}(N) = \frac{1}{n} Q^{(1)}(N^{\otimes n})$

The LSD thm:

$Q(N) = \sup_n Q^{(n)}(N)$

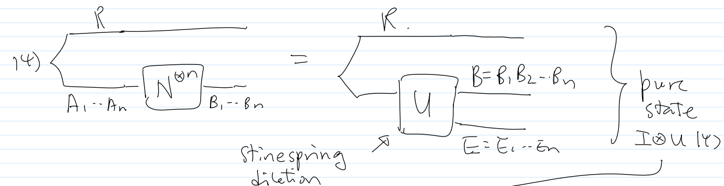
In other words:



$I_R \otimes N^{\otimes n}(|\psi\rangle\langle\psi|)$.

- Calculate $S(B_1\cdots B_n) - S(RB_1\cdots B_n)$
- divide by n
- sup over n.

So, what is $I_c(R > B_1\cdots B_n)$?



$I_c(R>B) = S(B) - S(BR) = S(ER) - S(E)$

$= \frac{1}{2}\Big([S(B) - S(BR) + S(R)] + [S(ER) - S(E) - S(R)]\Big)$

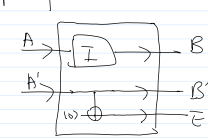
$= \frac{1}{2}[S(B{:}R) - S(E{:}R)]$

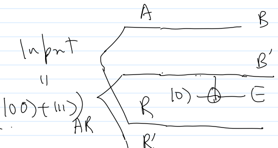
When transmitting classical data, we max $S(B{:}R)$ (evaluated on $\underbrace{\sum_x p_x |x\rangle\langle x|}_{R} \otimes \underbrace{N^{\otimes n}(\rho_x)}_{B}$)

For quantum data

not only we need to max Bob correlation with R, he needs to take out a portion of his data that's correlated with Eve. The QMI counts all correlation (classical + quantum) and the classical portion is quantified by $S(R{:}E)$.

eg. let N = 1 qbit + 1 cbit. Stinespring's dilation





$(|00\rangle + |11\rangle)_{AR}$

$(|00\rangle + |11\rangle)_{A'R'}$

Output $= \underbrace{(|00\rangle + |11\rangle)_{RB}}_{S(R:B)=2} \otimes \underbrace{(|000\rangle + |111\rangle)_{R'B'E}}_{S(R':B')=1,\ S(R':E)=1}$

$S(RR'{:}BB') = 3$, $S(RR'{:}E) = 1$

$I_c(RR' > BB') = \frac{1}{2}(3-1) = 1$ (expected).

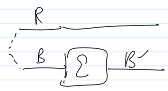
Properties of the coherent information $I(R\rangle B)$:

① Invariant under local unitary on R & B

② Invariant under attaching ancillas in pure states

③ Non increasing under local operation on B's side

Pf by monotonicity of QMI:



$$S(R:B)_\rho \geq S(R:B')_{I\otimes\mathcal{E}(\rho)}$$

$$S(R) + S(B) - S(RB) \qquad S(R) + S(B') - S(RB')$$

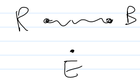
same, since evaluated on $\mathrm{tr}_B\,\rho = \mathrm{tr}_{B'}[I\otimes\mathcal{E}(\rho)]$

$$\therefore\ I(R\rangle B)_\rho \geq I(R\rangle B')_{I\otimes\mathcal{E}(\rho)}.$$

quantum data processing inequality

④ Discarding a system on R may increase/decrease $I(R\rangle B)$.

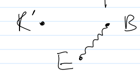
eg1.



Initial state $= (|00\rangle+|11\rangle)_{RB}|0\rangle_E$

$$I(R\rangle B) = \underset{①}{S(B)} - \underset{⓪}{S(RB)} = 1$$

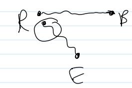
Discarding R (giving it to E)



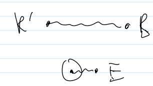
final state $= |0\rangle_R(|00\rangle+|11\rangle)_{BE}$

$$I(R'\rangle B) = \underset{①}{S(B)} - \underset{①}{S(R'B)} = 0.$$

eg2.



$$I(R\rangle B) = S(B) - S(BR) = 1 - 1 = 0$$



$$I(R'\rangle B) = 1 - 0 = 1.$$