Lec 8, June 1, 2010

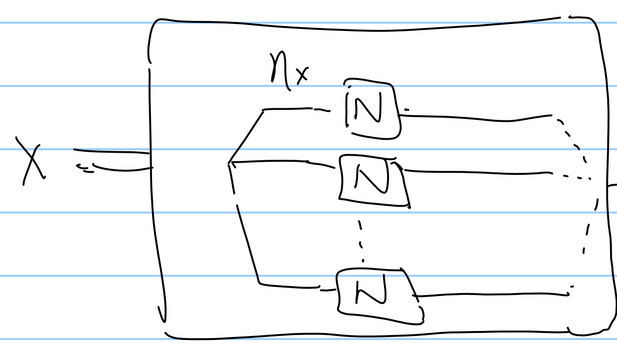
Note Title 31/05/2010

Last time:

$$C(Q) = \max_{p_x} \chi(\{p_x, \rho_x\}) \quad \text{where} \quad x \longrightarrow \boxed{Q} \longrightarrow \rho_x$$

$$C(\mathcal{N}) = \max_{p_x, \eta_x} \frac{1}{n} \chi\left(\{p_x, \mathcal{N}^{\otimes n}(\eta_x)\} \right)$$

interpreted as:



$\rho_x = \mathcal{N}^{\otimes n}(\eta_x)$

HSW Thm (after Holevo, Schumacher & Westmoreland)

Proved $\approx$ 1997.

Convenient to define

$$\chi(N) = \chi^{(1)}(N) = \max_{\{p_x, \rho_x\}} \chi\Big(\underbrace{\{p_x, N(\rho_x)\}}_{\text{ensemble received by Bob after 1 use of channel}}\Big),$$

$$\chi^{(n)}(N) = \max_{\{p_x, \rho_x\}} \frac{1}{n} \chi\left(N^{\otimes n}\right)$$

Thus $C(N) = \sup_n \chi^{(n)}(N) \geq \chi^{(n)}(N) \geq \chi^{(1)}(N)$

Consequences:

① $C(N) = 0$ iff $\chi^{(1)}(N) = 0$

$$\chi^{(1)}(N) = \text{QMI of } \left[\sum_x p_x |x\rangle\langle x| \otimes N(\rho_x)\right]$$

Recall $S(A:B)=0$ iff A, B in a product state

$\therefore C(N)=0$ iff $N(\rho_x)$ independent of x

for any choice of $\{p_x, \rho_x\}$

iff $N(\rho)=\rho_0$ const

N completely useless

(Bob can simulate N by creating ρ_0 on his side!)

(2) We can prove continuity of $C(N)$ (as a function of N).

(Tricky for a regularized expression, but doable.)

So similar channels have similar capacities.

$\boxed{\text{later in the course}}$

(3) $\chi^{(1)}(N)$ is a useful lower bound to $C(N)$.

Will calculate this for several channels (this lecture).

(4) For some special channels, $\chi^{(1)}(N) = \chi^{(n)}(N)$!

So we know $C(N)$ for these channels (this lecture).

(5) Up until 2008, the hope was that $\chi^{(1)}(N) = \chi^{(n)}(N)$ for ALL channels, but Hastings gave a counterexample.

Will discuss these "additivity conjectures" (this lecture)

July 8th - Fukuda will give a guest lecture on the ideas behind Hasting's counterexample :)

Finding $\chi^{(1)}(\mathcal{N}) = \max_{\{p_x, \rho_x\}} \chi(\{p_x, \mathcal{N}(\rho_x)\})$

$$= \max_{\{p_x, \rho_x\}} \underbrace{S\Big(\sum_x p_x \mathcal{N}(\rho_x)\Big)}_{\text{max this entropy}} - \sum_x p_x \underbrace{S(\mathcal{N}(\rho_x))}_{\text{min these entropies}}$$

Reference: 9912122 Schumacher & Westmoreland

① ρ_x can be chosen to be pure (elementary argument)

② Suffices to have d^2 states in the ensemble
where d = dim of input to $\mathcal{N}$
(argument similar to those in Davies, IEEE TIT 24 (1978) 596).

eg1. $\mathcal{N}$ = d-dim erasure channel

$$\mathcal{N}(\rho) = (1-p)\,\rho + p\,|E\rangle\langle E| \quad \text{for } |E\rangle \text{ error symbol}$$

Consider any input ensemble $\{p_x, \rho_x\}$ (ρ_x pure):

$$\forall x,\ \mathcal{N}(\rho_x) = (1-p)\rho_x + p\,|E\rangle\langle E| \ \text{ and } \ \boxed{S(\mathcal{N}(\rho_x)) = H(p)} \quad (1)$$

no need to min

$$\text{Average output state} = \sum_x p_x \mathcal{N}(\rho_x) = \mathcal{N}\Big(\sum_x p_x \rho_x\Big)$$

$$= (1-p)\Big(\sum_x p_x \rho_x\Big) + p\,|E\rangle\langle E|$$

$$\boxed{S\Big(\sum_x p_x \mathcal{N}(\rho_x)\Big) = (1-p)\, S\Big(\sum_x p_x \rho_x\Big) + \cancel{p\, S(|E\rangle\langle E|)} + H(p)} \quad (2)$$

max by $\sum_x p_x \rho_x = \frac{I}{d}$ eg. $p_x = \frac{1}{d}$, $\rho_x = |x\rangle\langle x|$

$$\therefore\ \chi^{(1)}(\mathcal{N}) = \max_{\{p_x,\rho_x\}} \chi(\{p_x, \mathcal{N}(\rho_x)\}) = (1) - \sum_x p_x (2) = (1-p)\log d. \quad \text{(cf classical case)}$$

eg2. N = d-dim depolarizing channel

$$N(\rho) = (1-p)\rho + p\frac{I}{d}$$

For any pure state ensemble $\{p_x, \rho_x\}$

$$\boxed{S(N(\rho_x)) = \text{Entropy of } \left\{ \left(1-p+\frac{p}{d}\right), \frac{p}{d}, \ldots \frac{p}{d} \right\}} \quad (1)$$

again independent of ρ_x, no need to minimize.

$$\sum_x p_x N(\rho_x) = (1-p)\left(\sum_x p_x \rho_x\right) + p\frac{I}{d}$$

Again $\underbrace{S\left(\sum_x p_x N(\rho_x)\right)}$ max by $\sum_x p_x \rho_x = \frac{I}{d}$

$p_x = \frac{1}{d}$, $\rho_x = |x\rangle\langle x|$

$$\chi^{(1)}(N) = \log d - (1)$$

$$= \log d - \left(1-p+\frac{p}{d}\right)\log\left(1-p+\frac{p}{d}\right) - \left(\frac{d-1}{d}\right)p\left[\log\frac{p}{d}\right] \quad \ldots$$

When $d=2$, $\chi^{(1)}(N) = 1 - H(\frac{p}{2})$ (cf Classical BSC)

eg3 $\underset{\varphi}{N}(\rho) = 0.8\rho + 0.15 X\rho X + 0.05 Z\rho Z$

2 dim input & output, X, Z : Pauli's.

① $S(N(\rho))$ minimized by $\rho = |+\rangle\langle+|$ or $|-\rangle\langle-|$.

$S(N(|\underline{+}\rangle\langle\underline{+}|)) = H(0.05)$

② $S(\sum_x p_x N(\rho_x))$ maximized by $\sum_x p_x N(\rho_x) = \frac{I}{2}$

Since $\sum_x p_x N(\rho_x) = N(\sum_x p_x \rho_x)$ and $N(I) = I$ (unital)

choose $\rho_0 = |+\rangle\langle+|$ & $\rho_1 = |-\rangle\langle-|$ with equal probs

$\therefore \chi^{(1)}(N) = 1 - H(0.05)$

In general, if $\mathcal{N}$ unital & there is a set of states $|\psi_i\rangle$

each $|\psi_i\rangle$ minimized $S(\mathcal{N}(|\psi_i\rangle\langle\psi_i|))$ and

the convex hull of $|\psi_i\rangle\langle\psi_i|$ includes $\frac{I}{d}$

ie $\exists\, p_i$'s s.t. $\sum_i p_i |\psi_i\rangle\langle\psi_i| = \frac{I}{d}$

then $\{p_i, |\psi_i\rangle\}$ is an optimal ensemble for $\chi^{(1)}(\mathcal{N})$.

Qn: are orthogonal fx always optimal?
They're afterall the most distinguishable inputs.

Ans: No .... by Fuchs PRL 79 (1997) 1162.
There, a channel was cooked up for the demonstration
It turns out the case for the amplitude damping
channel also (9912122).

These Nonorthogonal signal states are considered a
"Quantum" advantage...

eg 4. Amplitude damping (AD) channel

$$N(\rho) = A_0 \rho A_0^\dagger + A_1 \rho A_1^\dagger$$

$A_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix}$	$A_1 = \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix}$
$A_0(a\|0\rangle + b\|1\rangle)$ $= a\|0\rangle + \sqrt{1-\gamma}\,\|1\rangle$	$A_1(a\|0\rangle + b\|1\rangle)$ $= \sqrt{\gamma}\, b\|0\rangle$.

$|0\rangle, |1\rangle$ are energy eigenstates and $|1\rangle$ is at a higher energy level than $|0\rangle$.

AD represents the physical process of deexcitation.

Consider $\gamma = \frac{1}{2}$.

- Restricting ρ_x to orthogonal states: $|\psi_0\rangle$, $|\psi_1\rangle$

  $\chi(\{p_x, \rho_x\})$ max at $p_0 = p_1 = \frac{1}{2}$, $\rho_0 = |+\rangle\langle+|$, $\rho_1 = |-\rangle\langle-|$

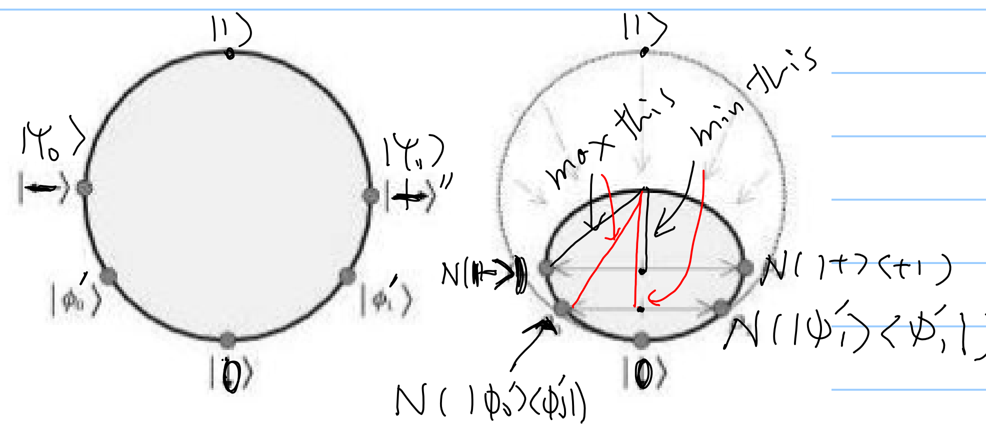
  and $\chi = 0.4567$

- If instead, $|\psi_0'\rangle = a|0\rangle + b|1\rangle$, $|\psi_1'\rangle = a|0\rangle - b|1\rangle$

  for a slightly larger than b ($\langle\psi_0'|\psi_1'\rangle \approx \cos 80°$)

  $\chi = 0.4717$

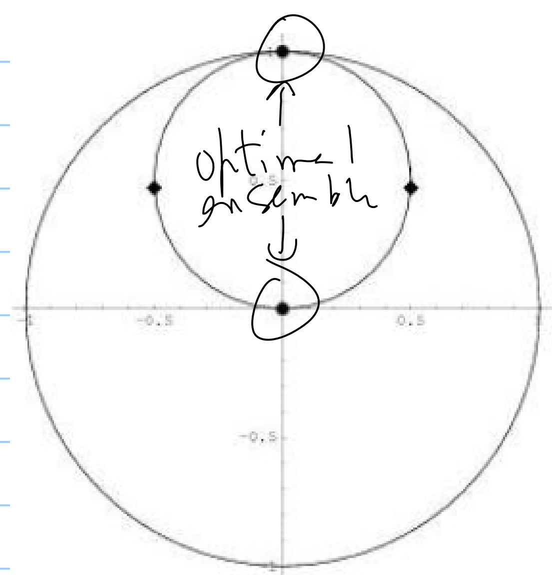
Intuition: "$|0\rangle$" state suffers less AD.

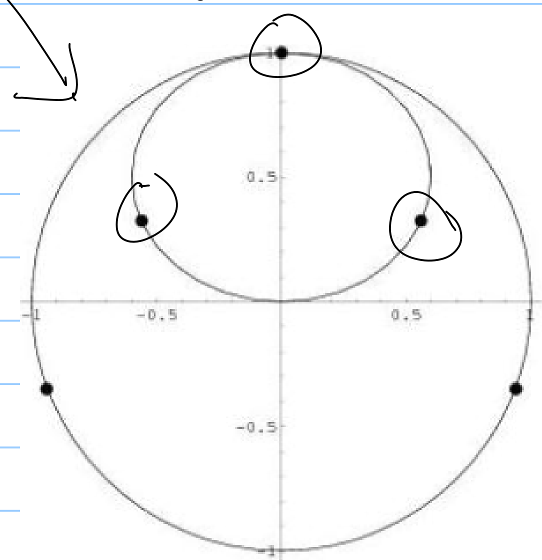


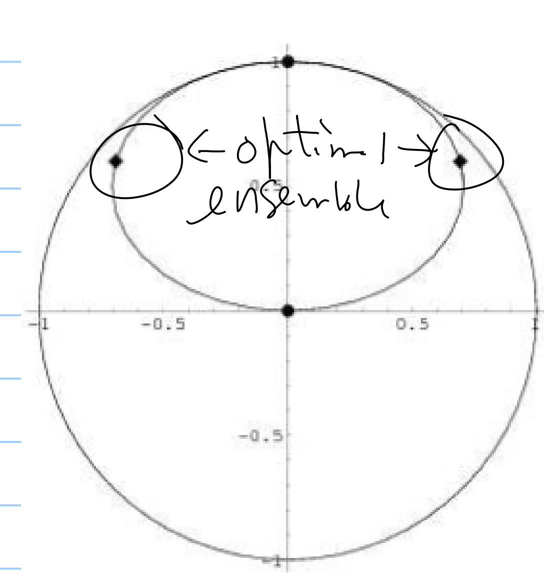
Qn: for d=2, will we need 3 or 4 states in the optimal ensemble?

0109079 King, Nathason, & Ruskai:

channel requiring 3 states







① Depolarizing + shift

② AD channel $\gamma = \frac{1}{2}$

4 states needed in 0403176

Qn: How hard is it to calculate $\chi^{(1)}(N)$?

Ans: (Beigi & Shor 0709.2090)

Let $c \in \mathbb{R}$. Deciding whether $\chi^{(1)}(N) > c$

or $\chi^{(1)}(N) < c - \varepsilon$ for $\varepsilon = \frac{1}{\mathrm{poly}(d)}$

(d = dim of input) is NP complete.

NB this says little about hardness of estimating $\chi^{(n)}$ since tensor product channels $N^{\otimes n}$ can be easier to handle.

Back to $C(N)$:

We only know $C(N)$ if $\chi^{(1)}(N) = \chi^{(n)}(N)$.

Def: We say χ is <u>strongly</u> additive if

$\forall N_1, N_2$, $\chi^{(1)}(N_1 \otimes N_2) = \chi^{(1)}(N_1) + \chi^{(2)}(N_2)$

Def: We say χ is <u>weakly</u> additive if

$\forall N$, $\chi^{(n)}(N) = \chi^{(1)}(N)$

NB: strongly additive $\Rightarrow$ weakly additive

NB: "$\geq$" easily proved. The question is whether it can be $\gneq$. If we restrict to product states on the LHS, then $=$.

Known additivity results:

① If N_1 is entanglement breaking

then $\forall N_2$, $\chi^{(1)}(N_1 \otimes N_2) = \chi^{(1)}(N_1) + \chi^{(1)}(N_2)$

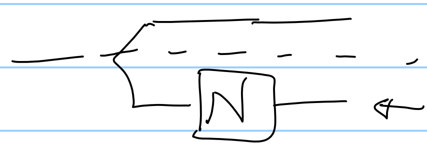
(0302031?)

Cor: If N is entanglement breaking,

then $C(N) = \chi^{(1)}(N)$.

Def: N is entanglement breaking

if $\forall \rho_{AB}$, $I \otimes N(\rho_{AB})$ is separable.



← not entangled with anything.

Fact: N ent breaking

iff $N =$ "QCQ" channel

ie $\to \boxed{N} \to \;=\; \to \boxed{\boxed{M} \xrightarrow{x} \boxed{Q} \xrightarrow{\rho_x}} \to$

Intuition: superadditivity of χ is due to entanglement between the inputs to the 2 channels.

So this is expected ...

Also, classical channels are entanglement breaking.
So we recover Shannon's noisy coding theorem & the single letter capacity expression.

(2) If N_1 has 2-dim inputs & outputs & <u>N_1 unital</u>

then $\forall N_2$, $\chi^{(1)}(N_1 \otimes N_2) = \chi^{(1)}(N_1) + \chi^{(2)}(N_2)$

Cor: If N_1 is a qubit unital channel, then $C(N) = \chi^{(1)}(N)$

(King 0103156)

NB: These include eg 2 & 3 (so our calculation of $\chi^{(1)}(N)$ gives $C(N)$ as well!).

Fact: N_1 is a qubit unital channel

iff $\to \boxed{N_1} \to \; = \; \to \boxed{U} - \boxed{\mathcal{E}} - \boxed{V} \to$

where U, V unitary & $\mathcal{E}(\rho) = \sum_i p_i \sigma_i \rho \sigma_i$ ← Pauli's

③ If N_1 = d-dim depolarizing channel, then

$\forall N_2$, $\chi^{(1)}(N_1 \otimes N_2) = \chi^{(1)}(N_1) + \chi^{(1)}(N_2)$

⋮

Same consequences as before

④ For N = erasure channel,

I think $C(N) = \chi^{(1)}(N) = (1-p)\log d$.

⑤ $\chi^{(1)}(N_1 \otimes N_2)$ additive $\forall$ channels $N_1 N_2$

iff — — — — — $\forall N_2$ & $\forall N_1$ unital.

(Fukuda 0608010)

6 $\chi^{(1)}$ strongly additive ($\forall$ channels)

$\Updownarrow$

$\chi^{(1)}$ weakly additive ($\forall$ channels).

7 $\exists\ N_1, N_2$ s.t.

$$\chi^{(1)}(N_1 \otimes N_2) \gneq \chi^{(1)}(N_1) + \chi^{(2)}(N_2).$$

Hastings 0809.3972.

So, how bad is the nonadditivity? we don't know.

But we don't know $C(N)$ for $N =$ AD channel

In fact, for γ small, we have no upper bound to the capacity except for $C(N) \leq 1$!

(Eisert told me something last week ....).