Further reason to hope $\chi^{(1)}(N) = \chi^{(n)}(N)$

Equivalences of additivity conjectures:

Def entanglement of formation:

ent of pure state $|\psi_i\rangle$

$$E_f(\rho) = \min_{\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|} \sum_i p_i E(|\psi_i\rangle)$$

$= S(\mathrm{tr}_A |\psi_i\rangle\langle\psi_i|)$

$=$ entropy of $|\text{Schmidt coeffs}|^2$

Average entanglement in a pure state decomposition

given

you choose these, ρ, $|\psi_i\rangle$: states on $\mathcal{H}_A \otimes \mathcal{H}_B$

Known: Entanglement dilution takes $\approx n E(|\psi_i\rangle)$ ebits & $O(\sqrt{n})$ cbits to create n copies of $|\psi_i\rangle$

- Intuitive protocol to make $\rho^{\otimes n}$ using ebits & cbits:

Alice & Bob can use $\approx n E_F(\rho)$ ebits to create $|\psi_i\rangle^{\approx \otimes n p_i}$ for each i. Then Alice draws from $\{p_i\}$ iid n times, gets p_{i_1} p_{i_2} ... p_{i_n}, tells Bob these outcomes and they place $|\psi_{i_1}\rangle$ in A_1B_1, $|\psi_{i_2}\rangle$ in A_2B_2, ... $|\psi_{i_n}\rangle$ in A_nB_n and <u>forget</u> $p_{i_1} \ldots p_{i_n}$. They then share $\rho^{\otimes n}$.

Thus the name "ent of formation". (That was $\approx$ 1996)

For a while, $E_F(\rho)$ was thought to be the <u>exact</u> no more no loss <u>asymptotic</u> # ebits needed per copy of ρ made.

In 0008134, Hayden, Horodecki & Terhal defined the entanglement cost:

$$E_C(\rho) = \lim_{\substack{n\to\infty \\ \varepsilon_n \to 0}} \frac{1}{n} \left(\begin{array}{l} \#\text{ ebits needed to approx } \rho^{\otimes n} \\ \text{with accuracy } \varepsilon_n \text{ w/ free cbits} \end{array} \right)$$

& proved:

$$E_C(\rho) = \lim_{n\to\infty} \frac{1}{n} E_F(\rho^{\otimes n}) \underset{\uparrow}{\leq} E_F(\rho)$$

$\neq$ if there is a better pure state decomp over $A_1A_2\cdots A_n$ vs $B_1\cdots B_n$!

For a while, everyone hoped $E_F(\rho) = E_C(\rho)$.

In 0305035, Shor proved that the following are either all true or all false:

① $\forall N_1, N_2,\quad \chi^{(1)}(N_1 \otimes N_2) = \chi^{(1)}(N_1) + \chi^{(1)}(N_2)$

② $\forall \rho_1, \rho_2,\quad E_F(\rho_1 \otimes \rho_2) = E_F(\rho_1) + E_F(\rho_2)$

③ $\forall N_1, N_2,\quad S_{min}(N_1 \otimes N_2) = S_{min}(N_1) + S_{min}(N_2)$

(where $S_{min}(N) = \min_\rho S(N(\rho))$ = min output entropy).

④ $\forall \rho_{A_1 B_1 A_2 B_2},\quad E_F(\rho_{A_1A_2|B_1B_2}) \geq E_F(\rho_{A_1B_1}) + E_F(\rho_{A_2B_2})$