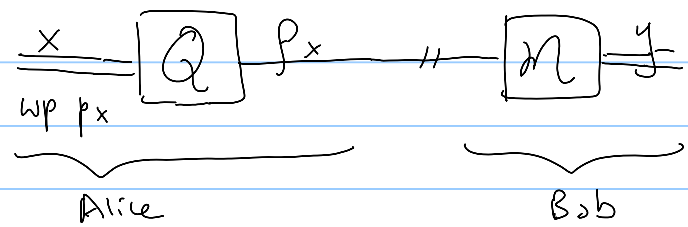
Note Title 24/05/2010

Lec 6, May 25, 2010, ctd May 27

Asymptotic classical communication capacity of quantum states & channels.

Recall given an ensemble $\mathcal{E} = \{p_x, \rho_x\}$



Treating the x->y process as a classical channel, the capacity is Iacc of the ensemble.

But we can do better given the Q box -- not because of Alice but because of Bob!

We can do even better given a channel --

$$I_{acc}(\mathcal{E}) = \max_{M} I(X:Y) \leq \chi(\mathcal{E}) = S\Big(\sum_x p_x \rho_x\Big) + \sum_x p_x S(\rho_x)$$

Q1 How much can Alice comm to Bob if

① She decides what "x" instead of drawing $x \sim p(x)$

② She can use $\boxed{Q}$ many times? (Bob measures collectively)

Q2 Same question, but a channel $\mathcal{N}$ (instead of $\boxed{Q}$) is available instead?

Results:

$$C(N) = \sup_n \frac{1}{n} \chi(\eta^{\otimes n})$$

where $\chi(\eta) = \max_{\mathcal{E}=\{p_x, \rho_x\}} \chi(\eta(\mathcal{E}))$

Direct coding uses the fact:
Asymptotic comm rate of Q
$= \max_{p_x} \chi(\{p_x, \rho_x\})$

Converse uses:
- upper bound on # outcomes in optimal measurements
- classical Fano's ineq

Uses random quantum codewords + pretty good measurements

$\Downarrow$

Conditional typicality $\Rightarrow$ Conditions for gentle meas lemma & packing lemma

$\Uparrow$ (pretty good measurements)

1 Pretty good measurement

P. Hausladen '93 and W. K. Wootters, J. Mod. Optics, 41, 2385 (1994).

Suppose a Q system is prepared in one of the possible states $\rho_1\ \rho_2 \dots \rho_K$.

The PGM is given by the POVM:

$$M_i = \Lambda^{-\frac{1}{2}} \rho_i \Lambda^{-\frac{1}{2}} \quad \text{for } i = 1, \dots K$$

$$M_{K+1} = I - \sum_{i=1}^{K} M_i$$

$\Lambda = \sum_{i=1}^{K} \rho_i$, and $\Lambda^{-\frac{1}{2}}$ performed only on supp (Λ).

Note that the PGM is still well defined if $\rho_i \geq 0$ but otherwise unconstraint (OK if $\operatorname{tr} \rho_i \neq 1$ or $\rho_i \nleq I$).

Elaborating the notations:

Since $\Lambda = \sum_{i=1}^{K} \rho_i$ is positive semidef.

let $\Lambda = \sum_j \lambda_j |\phi_j\rangle\langle\phi_j|$ be its spectral decomposition

$\Lambda^{-\frac{1}{2}} = \sum_j \lambda_j^{-\frac{1}{2}} |\phi_j\rangle\langle\phi_j|$, $I_{supp(\Lambda)} = \sum_j |\phi_j\rangle\langle\phi_j|$

Thus:

$M_i \geq 0$, $\sum_{i=1}^{K} M_i = \Lambda^{-\frac{1}{2}} \Lambda \Lambda^{-\frac{1}{2}} = I_{supp(\Lambda)}$

$M_{K+1} = I - I_{supp(\Lambda)} = I_{complement(supp(\Lambda))}$.

$\forall i,\ 0 \leq tr(\rho_i M_{K+1}) \leq tr(\Lambda M_{K+1}) = 0$ (;, =)

Intuitively, the measurement has an error if the state is ρ_i but the outcome is $j \neq i$.

How good is the "pretty good" meas?

- If ρ_i's are orthogonal, it is perfect.
- Upon "googling" many hits on PGM being optimal in specific applications.
- For pure states $\rho_i = |\psi_i\rangle\langle\psi_i|$ given equiprobably, ave prob error $\leq \frac{1}{K} \sum_{i \neq j} |\langle\psi_i|\psi_j\rangle|^2$

We'll use packing lemma to bdd error prob.

(2) Gentle measurement lemma (Winter ...):

Let $\rho \geq 0$, $tr(\rho) \leq 1$, $0 \leq E \leq I$

If $Tr(\rho M) \geq 1-\varepsilon$

then $\exists U$ s.t. $\| \underbrace{U E^{\frac{1}{2}} \rho E^{\frac{1}{2}} U^{\dagger}}_{\text{best possible post meas state for the outcome corr to } E} - \rho \|_{tr} \leq \sqrt{8\varepsilon}$

Interpretations:

Given a state ρ, if meas yields 1 outcome whp

then the state is hardly changed by the meas.

(3) Def: for a set of states $S = \{\rho_1, \rho_2, \ldots \rho_k\}$

let distinguishability error of S be

$$d_e(S) = \min_{\text{meas}} \operatorname{ave}_i \Pr(\text{outcome} = j \mid \text{state} = \rho_i)$$

NB can upper bound $d_e(S)$ by considering specific measurements.

(4) The packing lemma:

Let $p_x = \text{prob}(x)$, ξ_x states, $\xi = \sum_x p_x \xi_x$

If projectors Π, Π_x exist s.t. $\forall x$:

1. $\text{Tr}(\xi_x \Pi) \geq 1-\varepsilon$
2. $\text{Tr}(\xi_x \Pi_x) \geq 1-\varepsilon$
3. $\text{Tr}(\Pi_x) \leq d_1$
4. $\Pi \xi \Pi \leq \frac{\Pi}{d_0}$

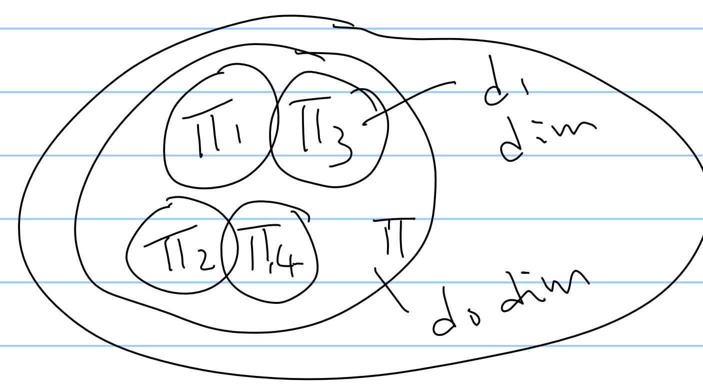
tve fraction

☆ 5. $S = \{\gamma_1 \ldots \gamma_K\}$, each $\gamma_x = \xi_x$ wp p_x (iid), $K = \frac{d_0}{d_1} f$

Then $\mathbb{E}_S\, de(S) \leq 2(\varepsilon + \sqrt{8\varepsilon}) + 4f$ (achieved w/ "PGM").

What does this lemma mean?

Conditions (2) & (3) say each ξ_x lives in some d_1-dim space (up to ε approx) defined by ξ_x. Condition (3) says all ξ_x lives in a space defined by Π.



Condition (4) says $\Pi \rho \Pi \leq \frac{\Pi}{d_0}$

Since $\mathrm{tr}(\rho_x \Pi) \geq 1-\epsilon$, $\Pi \rho_x \Pi \approx \rho_x$

& $\Pi \rho \Pi \approx \rho$

$\therefore$ Condition (4) bounds the max eigenvalue of ρ to be no more than $\frac{1}{d_0}$, or $d_0 \approx \frac{1}{\lambda_{max}(\rho)}$

where $\lambda_{max}(\Gamma)$ denotes the max eigenvalue of Γ.

In general: for 2 rank pure states $|\psi\rangle$ & $|\phi\rangle$

$\lambda_{max}(|\psi\rangle\langle\psi| + |\phi\rangle\langle\phi|)$ is large when

$|\psi\rangle$ & $|\phi\rangle$ have high overlap.

In the current problem: $\rho = \sum_x p_x \rho_x$

We expect $\lambda_{max}(\rho)$ to be small if the ρ_x's are distinguishable but having mixed state ρ_x complicates things.

eg $\rho_1 = \left(\frac{1}{4} |0\rangle\langle 0| + \ldots \right)$

$\rho_2 = \left(\frac{1}{4} |0\rangle\langle 0| + \ldots \right)$

Here ..... are smaller terms that do not enter the calculation of $\lambda_{max}(\rho_1 + \rho_2)$

None the less, ... still affect their distinguishability.

This problem is worse if rank$(\rho_{1,2})$ are large.

The packing lemma tells us, if we're communicating using quantum states ρ_x and we know little about them except each lives under Π_x of dim d_1, then we can send $k = \frac{d_0}{d_1} f$ message w/o much error

α — the fraction of space we're willing to leave blank

this comes from how distinguishable ρ_x's are.

(Also $d_0 \geq \text{rank}(\Pi)$, $d_1 \approx$ size of ρ_x i.e. $\frac{d_0}{d_1}$ sounds right.)

eg. $|\psi_0\rangle = \sqrt{1-p}\,|0\rangle + \sqrt{p}\,|1\rangle$, $\rho_0 = |\psi_0\rangle\langle\psi_0|$, $p_0 = \frac{1}{2}$

$|\psi_1\rangle = \sqrt{1-p}\,|0\rangle - \sqrt{p}\,|1\rangle$, $\rho_1 = |\psi_1\rangle\langle\psi_1|$, $p_1 = \frac{1}{2}$

$$\rho = \begin{bmatrix} 1-p & 0 \\ 0 & p \end{bmatrix}$$

Choose $\Pi_0 = |\psi_0\rangle\langle\psi_0|$, $\Pi_1 = |\psi_1\rangle\langle\psi_1|$ $\therefore d_i = 1$

If we choose $\Pi = |0\rangle\langle 0|$ (when p large)

$\Pi \rho \Pi \leq \frac{\Pi}{d_0}$, $d_0 = 1$ Then $k = 1$

If we choose $\Pi = I$ (when $p \doteq \frac{1}{2}$)

$\Pi \rho \Pi \leq \frac{\Pi}{d_0}$ for $d_0 = 2$ Then $k = 2$

Proof: Let $\gamma_i = \rho_{x_i}$. Define a "PGM" using $\Pi\, \Pi_{x_i}\, \Pi$ & lower the expected prob of error.

$$\Lambda_i = \Pi\, \Pi_{x_i}\, \Pi, \quad \Lambda = \sum_{i=1}^{K} \Lambda_i,$$

$$M_i = \Lambda^{-\frac{1}{2}} \Lambda_i \Lambda^{-\frac{1}{2}}, \quad M_{K+1} = I - \sum_{i=1}^{K} M_i$$

$$d_e(S) \leq \frac{1}{K} \sum_i \operatorname{Tr} \gamma_i (I - M_i)$$

The $\Lambda^{-\frac{1}{2}}$ in M_i is nasty....

Use $I - M_i = I - \Lambda^{-\frac{1}{2}} \Lambda_i \Lambda^{-\frac{1}{2}} = I - \left(\Lambda_i + \sum_{j \neq i} \Lambda_j\right)^{-\frac{1}{2}} \Lambda_i \left(\Lambda_i + \sum_{j \neq i} \Lambda_j\right)^{-\frac{1}{2}}$

$$\leq 2\underbrace{(I - \Lambda_i)} + 4 \underbrace{\sum_{j \neq i} \Lambda_j}$$

sensible approxs to $I - M_i$ if $\Lambda^{-\frac{1}{2}} \approx I$

$$\boxed{\begin{array}{l} \text{op ineq:} \\ I - (X+Y)^{-\frac{1}{2}} X (X+Y)^{-\frac{1}{2}} \\ \leq 2(I - X) + 4Y \end{array}}$$

$$\boxed{\gamma_i = \rho_{x_i}\text{-r.v}}$$

Since $\gamma_i \geq 0$,

$$de(S) \leq \frac{1}{K}\sum_i \mathrm{Tr}\,\gamma_i(I - M_i)$$

$$\leq \frac{2}{K}\sum_i \underbrace{\mathrm{Tr}\,\gamma_i(I - \Lambda_i)}_{1 - \mathrm{Tr}(\rho_{x_i}\Pi\Pi_{x_i}\Pi)} + \frac{4}{K}\sum_i\left[\sum_{j\neq i}\mathrm{Tr}(\gamma_i\Lambda_j)\right]$$

Next page

$$1 - \mathrm{Tr}(\rho_{x_i}\Pi\Pi_{x_i}\Pi)$$

$$= 1 - \mathrm{Tr}(\Pi\rho_{x_i}\Pi\cdot\Pi_{x_i})$$

$$\leq 1 - \mathrm{Tr}(\rho_{x_i}\cdot\Pi_{x_i}) + \sqrt{8\varepsilon}$$

$$\leq \varepsilon + \sqrt{8\varepsilon} \quad (\text{indep of } i \ \therefore \ \tfrac{1}{K}\textstyle\sum_i \text{ cancel out})$$

└ by (2)

$\boxed{\begin{array}{l}\text{if } \|X-Y\|_{tr} \leq c \\ \text{then } \forall\, 0 \leq P \leq I \\ |\mathrm{tr}\, P(X-Y)| \leq c\end{array}}$

+ gentle meas + (1)

$\to$ from previous page, seeking a bound for $\sum_j \mathrm{Tr}(\gamma_i \Lambda_j)$ for $j \neq i$

Here use the fact $\gamma_i = \xi_x$ wp x_i drawn iid.

$$\underset{\rho_1\rho_2\cdots\rho_K\ (j\neq i)}{\mathbb{E}} \sum_j \mathrm{Tr}(\gamma_i \Lambda_j) = \sum_{j\ (j\neq i)} \mathrm{Tr}\Big[\big(\underbrace{\mathbb{E}\,\xi_{x_i}}_{=\xi}\big)\,\Pi\,\big(\mathbb{E}\,\Pi_{x_j}\big)\,\Pi\Big]$$

independent

$$\leq \sum_j \mathrm{Tr}(\Pi\,\xi\,\Pi)\,(\mathbb{E}\,\Pi_{x_j})$$

$$③ \quad \leq \sum_j \mathrm{Tr}\left(\frac{\Pi}{d_0}\right)(\mathbb{E}\,\Pi_{x_j}) \quad ④$$

$$= \mathbb{E} \sum_j \mathrm{Tr}\left(\frac{\Pi}{d_0}\right)\Pi_{x_j}$$

(rank $\leq d_1$, eigenvalues $= 0, 1$)

$$\leq \mathbb{E} \sum_j \frac{d_1}{d_0} = K\frac{d_1}{d_0} \leq f.$$

Back to earlier Qn:

Let $\boxed{Q}$ be a box that emits ρ_x to Bob if Alice inputs x.

For any n, consider an (M, n) code transmitting $\log M$ bits to Bob by n uses of Q.

Let P_e = worst prob error, $\mathbb{E} P_e$ expected prob error

R achievable if there are $(2^{nR}, n)$ codes w/ $P_e \to 0$

Call capacity of Q, $C_1(Q) = \sup R$ achievable.

Theorem: $C(Q) = \max_{p_X} \chi(\{p_X, \rho_X\})$

Again, need a direct coding proof & a converse proof.

Converse:

Consider the state $\displaystyle\sum_{x_1 x_2 \cdots x_n} \overbrace{p(x_1 x_2 \cdots x_n)}^{\text{arbitrary}} |x_1 \cdots x_n\rangle\langle x_1 \cdots x_n| \otimes \rho_{x_1} \cdots \otimes \rho_{x_n}$

system labels $X_1 X_2 \cdots X_n$ $\quad B_1 \cdots B_n$

$nR \leq I(X_1 \cdots X_n : B_1 \cdots B_n)$ ← Holevo info n-shot arbitrary probs

$= S(B_1 \cdots B_n) - S(B_1 \cdots B_n | X_1 \cdots X_n)$

$\leq \displaystyle\sum_{i=1}^{n} S(B_i) - \sum_{x_1 \cdots x_n} p(x_1 \cdots x_n) S(\rho_{x_1} \otimes \cdots \otimes \rho_{x_n})$

Need $B_1 \cdots B_n$ in product state $=$ $\displaystyle\sum_{i=1}^{n} S(B_i) - \sum_{x_1 \cdots x_n} p(x_1 \cdots x_n) \sum_{i=1}^{n} S(\rho_{x_i})$

$= \displaystyle\sum_{i=1}^{n} S(B_i) - \sum_{i=1}^{n} \underbrace{p(x_i)}_{\text{marginal}} S(\rho_{x_i}) \leq n\, I(X_i : B_i)$

arbitrary probs 1-shot Holevo info

Direct coding:

Recall in Shannon's noisy coding theorem

$$\left.\begin{array}{l} C_1 = X_{11}\ X_{12}\ \cdots\ X_{1n} \\ C_2 = X_{21}\ X_{22}\ \cdots\ X_{2n} \\ \vdots \\ C_M = X_{M1}\ X_{M2}\ \cdots\ X_{Mn} \end{array}\right\}$$

$X_{ij} \sim p_X$ iid

whp these are typical sequences

(prob of outcome $\approx 2^{-nH(X)}$)

Here, we demand C_i's be drawn randomly among strongly typical sequences (next page) and to send message i Alice inputs $x_{i1}, x_{i2} \cdots x_{in}$ into the n uses of $\boxed{Q}$.

$\therefore$ States to be distinguished by Bob:

$$\gamma_i = \rho_{x_{i1}} \otimes \rho_{x_{i2}} \otimes \cdots \otimes \rho_{x_{in}}$$

Strongly typical sequence:

For r.v X, a sequence $C = X_1 X_2 \cdots X_n$ is ε-strongly typical if the empirical dist^n of X observed in C is ε-close to $\{p_x\}$

say for $x=1$, $q(1) = \frac{1}{n}(\# X_i\text{'s equal to } 1)$

for $x=2$, $q(2) = \frac{1}{n}(\# X_i\text{'s} = 2)$

$\vdots$

and $\| q - p \|_1 = \sum_x | q(x) - p(x) | \leq \varepsilon$

HW: ε-strongly typical sequences are ε' typical e.g. let $\Omega = \{1, 2, 3, 4\}$, $p(a) = \frac{a}{10}$, $n = 20$

$c = 3\,3\,3\,4\,4\ \ 2\,1\,3\,2\,2\ \ 2\,4\,3\,4\,3\ \ 4\,2\,4\,4\,3$

$q(1) = 1/20$
$q(2) = 5/20$
$q(3) = 7/20$
$q(4) = 7/20$

$p(1) = 1/10$
$p(2) = 2/10$
$p(3) = 3/10$
$p(4) = 4/10$

$\|p - q\|_1 = 0.2$. So c is 0.2-strongly typical.

NB we focus on $n \gg |\Omega|$

Given $C = x_1 \cdots x_n$ ε-strongly typical

What do we know about $\gamma_c = \rho_{x_1} \otimes \cdots \otimes \rho_{x_n}$?

empirical prob

Up to reordering the n systems, it's $\bigotimes_x \rho_x^{\otimes n q(x)}$

For each x, by discussion leading to quantum data compression, there's a projector Π_x s.t.

$$\mathrm{Tr}(\Pi_x \rho_x^{\otimes n q(x)}) \geq 1 - \delta_1$$

$$\dim \Pi_x \leq 2^{-n q(x) (S(\rho_x) + \varepsilon_1)}$$

$\therefore$ Given C, let $\Pi_C = \bigotimes_x \Pi_x$ ← acting on the correct systems

$$\mathrm{tr}\left[\Pi_C \, \rho_{x_1} \otimes \cdots \otimes \rho_{x_n}\right] \geq 1 - \delta_2$$

$$\dim \Pi_C \leq 2^{-n\left[\sum_x q(x) S(\rho_x) + \varepsilon_2\right]}$$

e.g. when $C = 33344\ 21322\ 24343\ 42443$.

7 systems compressed by Π_4

state $\gamma_C = \rho_3 \otimes \rho_3 \otimes \rho_3 \otimes \rho_4 \otimes \rho_4 \otimes \cdots \otimes \rho_4 \otimes \rho_3$

compressed by Π_3

for $x=1$, Π_1 acts on sys 7

$x=2$, Π_2 acts on sys 6, 9, 10, 11, 17

$x=3$, Π_3 acts on sys 1, 2, 3, 8, 13, 15, 20

$x=4$, Π_4 acts on sys 4, 5, 12, 14, 16, 18, 19

More precisely,

for each n, for c, γ, Π_c defined earlier,

& let $\rho = \sum_x p_x \rho_x$, Π projector onto typical space of $\rho^{\otimes n}$

$\exists\ \varepsilon_n, \delta_n$ s.t.

① $\mathrm{Tr}(\gamma_c \Pi_c) \geq 1 - \delta_n$

② $\mathrm{Tr}(\Pi_c) \leq 2^{n[\sum_x p_x S(\rho_x) + \varepsilon_n]}$

proof outlined already

proof ex — ③ $\mathrm{Tr}(\gamma_c \Pi) \geq 1 - \delta_n$

④ $\Pi \rho^{\otimes n} \Pi \leq 2^{-n[S(\rho) - \varepsilon_n]} \Pi$

(and $\varepsilon_n, \delta_n \to 0$ as $n \to \infty$)

Finally, back to direct coding proof for $C(Q)$:

The M messages are encoded as: $\gamma_1, \gamma_2, \dots, \gamma_M$

where $\gamma_i = \rho_{x_{i1}} \otimes \rho_{x_{i2}} \otimes \cdots \otimes \rho_{x_{in}} =: \rho_{C_i}$

for $C_i = x_{i1} x_{i2} \cdots x_{in}$ a randomly chosen strongly typical sequence

The packing lemma now applies

ξ_x being ρ_C, C strongly typical

Π_x being Π_C

d_1 - ... $2^{n[\sum_x p_x S(\rho_x) + \varepsilon_n]}$

Π - ... Π

d_0 - ... $2^{n[S(\sum_x p_x \rho_x) + \varepsilon_n]}$

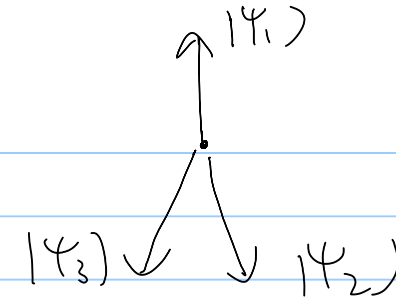
K ... M

$\therefore M$ can be $\frac{d_0}{d_1} \cdot f$

$\approx 2^{n[\chi(\{p_x, \rho_x\}) - \delta_3]}$

takes care of f & ε_n

eg. Consider the states



$$\rho_1 = |\psi_1\rangle\langle\psi_1|\, 0.9 + 0.1\, \frac{I}{2} \quad \text{for } |\psi_1\rangle = |0\rangle$$

$$\rho_2 = |\psi_2\rangle\langle\psi_2|\, 0.9 + 0.1\, \frac{I}{2} \quad \text{for } |\psi_2\rangle = \cos\theta\, |0\rangle + \sin\theta\, |1\rangle$$

$$\rho_3 = |\psi_3\rangle\langle\psi_3|\, 0.9 + 0.1\, \frac{I}{2} \quad \text{for } |\psi_3\rangle = \cos\theta\, |0\rangle - \sin\theta\, |1\rangle$$

s.t. $0.4\rho_1 + 0.3\rho_2 + 0.3\rho_3 = \frac{I}{2}$

So, the Q box emits one of $\rho_{1,2,3}$ on demand.

Applying what we proved, $C(Q) = \max_{p_1 p_2 p_3} \chi(\{p_i, \rho_i\})$

Since $S(\rho_i)$ same, $\chi(\{p_i, \rho_i\}) = S(\sum_i p_i \rho_i) - H((0.05))$.

max by $\sum_i p_i \rho_i = \frac{I}{2}$, or $p_1 = 0.4$, $p_2 = p_3 = 0.3$.

$=1$ $\qquad$ $=0.29$

What should Alice send to Bob?

Consider all strings of 1 2 3's of length n with $\approx 40\%$ 1's, 30% 2's, 30% 3's.

$H(\{0.4, 0.3, 0.3\}) \doteq 1.571$

There are $\approx 2^{n(1.571+\varepsilon)} \approx 2.971^n$ such strings.

Alice is drawing $2^{n(0.71-\delta)}$ strings from this pool (with replacement). If $\delta = 0.02$, there are $120 = M$ messages.

Say $C_1 = 2\,1\,1\,2\,3\,1\,2\,1\,3\,3$

$C_2 = 1\,3\,2\,1\,3\,2\,1\,2\,1\,3$

$\vdots$

$C_{120} = 2\,1\,3\,1\,2\,3\,1\,1\,2\,3$

$n = 10$ for illustration

Side Qn: lower bound Iacc for $n = 10$

So $\gamma_1 = \rho_2\,\rho_1\,\rho_1\,\rho_2\,\rho_3\,\rho_1\,\rho_2\,\rho_1\,\rho_3\,\rho_3$

$\gamma_2 = \rho_1\,\rho_3\,\rho_2\,\rho_1\,\rho_3\,\rho_2\,\rho_1\,\rho_2\,\rho_1\,\rho_3$

$\vdots$

$\gamma_{120} = \rho_2\,\rho_1\,\rho_3\,\rho_1\,\rho_2\,\rho_3\,\rho_1\,\rho_1\,\rho_2\,\rho_3$

product states, but now classically correlated once $C_1, C_2 \ldots C_{120}$ fixed.

Bob's measurement :

$$\boxed{\begin{aligned} &\Lambda_i = \Pi\, \Pi_{x_i} \Pi, \quad \Lambda = \sum_{i=1}^{K} \Lambda_i, \\ &M_i = \Lambda^{-\frac{1}{2}} \Lambda_i \Lambda^{-\frac{1}{2}}, \quad M_{K+1} = I - \sum_{i=1}^{K} M_i \end{aligned}}$$

e.g. $\gamma_{1000} = \rho_2 \rho_1 \rho_3 \rho_1 \rho_2 \rho_3 \rho_1 \rho_1 \rho_2 \rho_3$

Compress these 4 down to $\underbrace{2^{4\left(\underset{S(\rho_1)}{0.29} + \underset{\varepsilon}{0.1}\right)}}$ $\simeq 3$ dim $\left(S(\rho_1) = H(0.05) \doteq 0.29 \right)$

Compress to $2^{3(0.29+0.1)} \simeq$ "2.25" dims

$\therefore \Pi_{120} = \otimes$ of the 3 projections

Tr or rank of $\Lambda_{120} \approx$ "2.25"$^2 \times 3 \approx 14.5$

≈ 15 dims σ d_1

Theoritically, should be $2^{10(0.29+0.1)} \approx 15$ also.

Π projection onto typical space of $\left(\sum_i p_i \rho_i\right)^{\otimes 10}$

which is the full space. $d_0 \approx 2^{10}$

"Can send" $2^{10(\underbrace{1-0.29-0.2}_{\chi - 0.2})} \approx 120$ messages.

So $\bar{\Pi}\,\Pi_{120}\,\bar{\Pi} \approx \bar{\Pi}_{120}$.

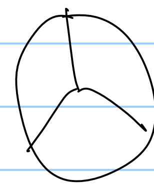
Same for $\bar{\Pi}_1 \cdots \bar{\Pi}_{119}$.

$$\Lambda = \sum_{i=1}^{120} \bar{\Pi}_i$$

highly collective

$\therefore \left\{ M_i = \Lambda^{-\frac{1}{2}}\, \bar{\Pi}_i\, \Lambda^{-\frac{1}{2}} \right\}$ defines a highly collective measurent

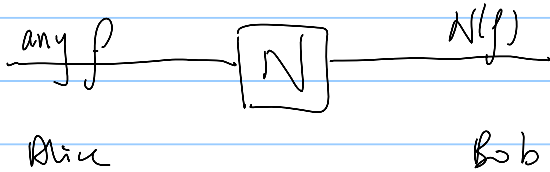
What is the I_{acc} of the ensemble $\mathcal{E} = \{p_i, \rho_i\}$ in the above example?

That of the  trine state ≈ 0.585

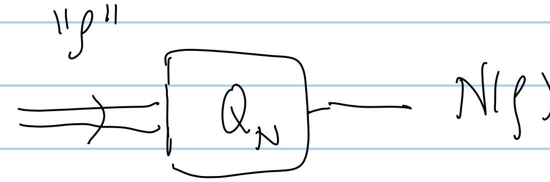
I_{acc} for the current ensemble should be even lower.

Now, classical capacity of a quantum channel.

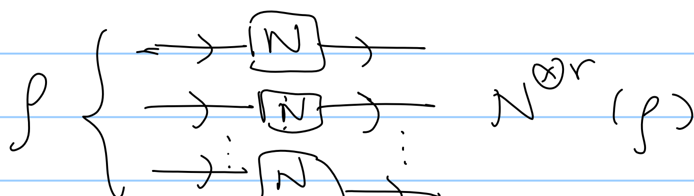
Basic use:



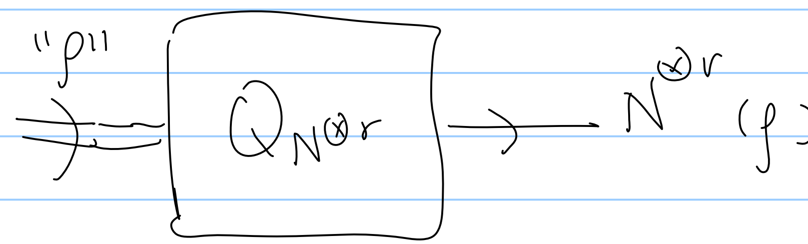
This is like our "Q" box:



Alice can also use $N^{\otimes r}$:



Corresponding Q box:



It follows from the capacity discussion of the Q box that

$$C(N) \geq \sup_r \frac{1}{r} \underbrace{\chi\left(N^{\otimes r}\right)}$$

and

$\leq$

defined as

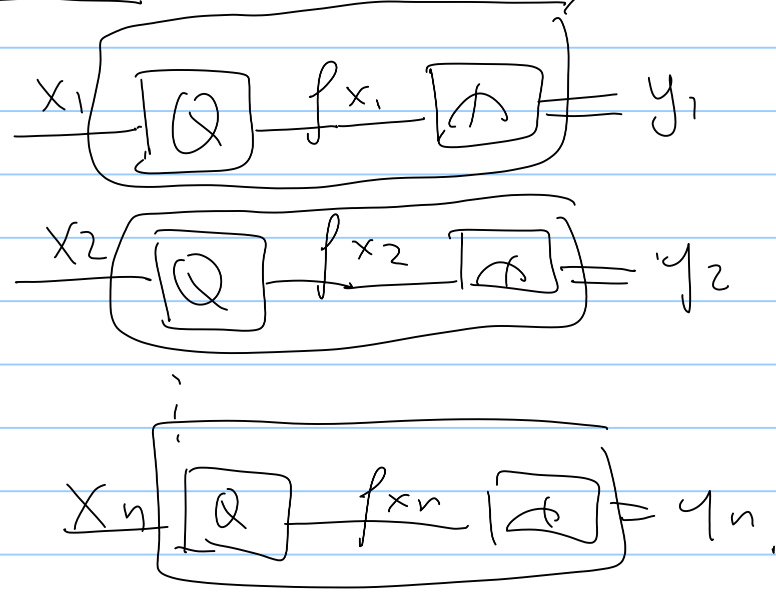
$$\max_{\{p_x, \rho_x\}} \chi\left(\{p_x, N^{\otimes r}(\rho_x)\}\right)$$

input to n channels.

This is called the HSW theorem, after Holevo, Schumacher & Westmoreland, PRA 56, 131 (1997), IEEE TIT 44, 269 (1998).

Protocol 1:

essentially classical channel $\mathcal{N}_0$

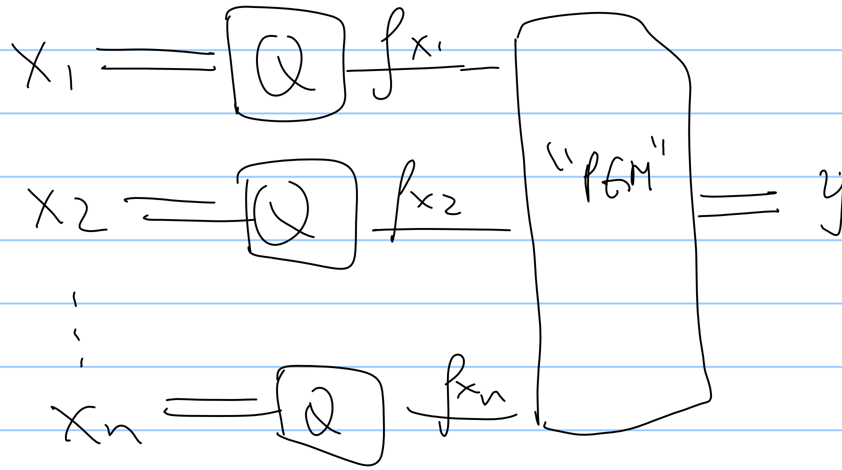


Alice chooses $p(x)$ to max $I_{acc}(\{p(x), \rho_x\})$

(The comm capacity of $\mathcal{N}_0$)

Protocol: Shannon's direct coding method for $\mathcal{N}_0$

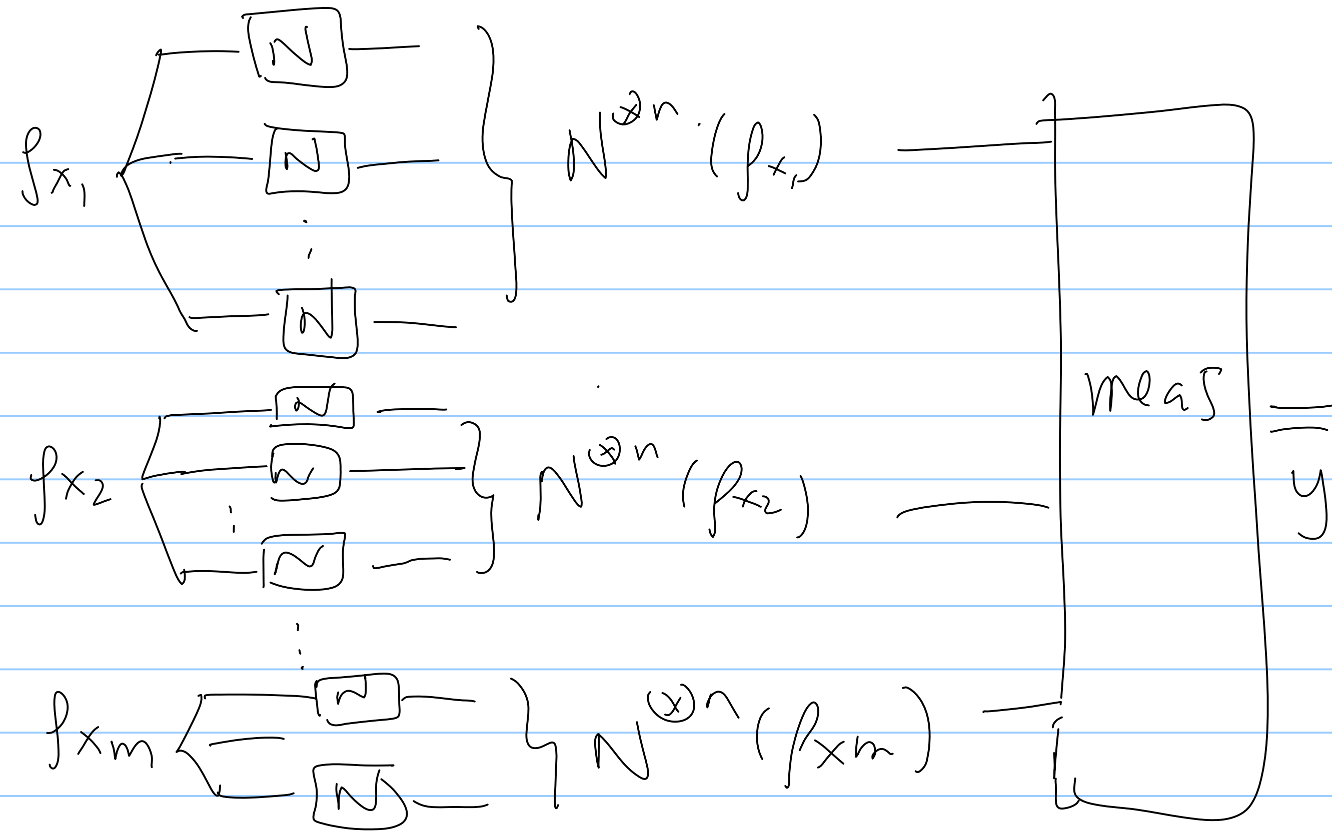
Protocol 2 :



Alice chooses $p(x)$ to max $\chi(\{p(x), \rho_x\})$
and this is the capacity $C(Q)$

Protocol : codewords randomly chosen from strongly typical sequences.

Protocol 3:



Max over $\{p_x, \rho_x\}$ ← n use input, n use output

$$\text{Capacity}: C(N) = \sup_n \frac{1}{n} \chi\left(p_x, N^{\otimes n}(\rho_x)\right)$$

Note that the capacity expression has an optimalization over r — called a "regularized" expression in contrast to the capacity expression for the classical channel or the Q box which involve only 1 copy of the resource (these are called single letterized expression).

The step $S(B_1 \cdots B_n) = S(B_1) + S(B_2) + \cdots + S(B_n)$ in the converse of capacity of Q-box breaks down when output state $N^{\otimes r}(\rho)$ is entangled in the channel setting.

To do:

- Some examples
- Additivity results vs non additivity (this really says where the problem comes from)
- brief discussion on the nonadditivity example

problems like not having upper bounds to capacity of AD channel