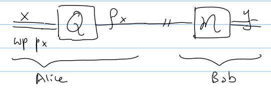
Lec 6, May 25, 2010, ctd May 27

Note Title 24/05/2010

Asymptotic classical communication capacity of quantum states & channels.

Recall given an ensemble $\mathcal{E} = \{p_x, \rho_x\}$



Treating the x->y process as a classical channel, the capacity is Iacc of the ensemble.

But we can do better given the Q box -- not because of Alice but because of Bob!

We can do even better given a channel --

$$I_{acc}(\mathcal{E}) = \max_{M} I(X:Y) \leq \chi(\mathcal{E}) = S\Big(\sum_x p_x \rho_x\Big) + \sum_x p_x S(\rho_x)$$

Q1 How much can Alice comm to Bob if

① She decides what "x" instead of drawing $x \sim p(x)$

② She can use Q many times? (Bob measures collectively)

Q2 Same question, but a channel N (instead of $\boxed{Q}$) is available instead?

Results:

$$\boxed{\begin{gathered} C(N) = \sup_n \frac{1}{n} \chi(\eta^{\otimes n}) \\ \text{where } \chi(\eta) = \max_{\mathcal{E} = \{p_x, \rho_x\}} \chi(\eta(\mathcal{E})) \end{gathered}}$$

Direct coding uses the fact: Asymptotic comm rate of Q $= \max_{p_x} \chi(\{p_x, \rho_x\})$

Converse uses:
- upper bound on # outcomes in optimal measurements
- classical Fanos ineq

Uses random quantum codewords + pretty good measurements

$\Downarrow$

Conditional typicality $\Rightarrow$ Conditions for gentle meas lemma & packing lemma $\Uparrow$

① Pretty good measurement

P. Hausladen '93 and W. K. Wootters, J. Mod. Optics, 41, 2385 (1994).

Suppose a Q system is prepared in one of the possible states $\rho_1\ \rho_2 \cdots \rho_K$.

The PGM is given by the POVM:

$$M_i = \Lambda^{-\frac{1}{2}} \rho_i \Lambda^{-\frac{1}{2}} \quad \text{for } i = 1, \ldots, K$$

$$M_{K+1} = I - \sum_{i=1}^{K} M_i$$

$$\Lambda = \sum_{i=1}^{K} \rho_i, \text{ and } \Lambda^{-\frac{1}{2}} \text{ performed only on } \mathrm{supp}(\Lambda).$$

Note that the PGM is still well defined if $\rho_i \geq 0$ but otherwise unconstraint (OK if $\mathrm{tr}\,\rho_i \neq 1$ or $\rho_i \not\leq I$)

Elaborating the notations:

Since $\Lambda = \sum_{i=1}^{K} \rho_i$ is positive semidef.

let $\Lambda = \sum_j \lambda_j |\phi_j\rangle\langle\phi_j|$ be its spectral decomposition

$$\Lambda^{-\frac{1}{2}} = \sum_j \lambda_j^{-\frac{1}{2}} |\phi_j\rangle\langle\phi_j|, \quad I_{\mathrm{supp}(\Lambda)} = \sum_j |\phi_j\rangle\langle\phi_j|$$

Thus:

$$M_i \geq 0, \quad \sum_{i=1}^{K} M_i = \Lambda^{-\frac{1}{2}} \Lambda \Lambda^{-\frac{1}{2}} = I_{\mathrm{supp}(\Lambda)}$$

$$M_{K+1} = I - I_{\mathrm{supp}(\Lambda)} = I_{\mathrm{complement}(\mathrm{supp}(\Lambda))}.$$

$$\forall i, \quad 0 \leq \mathrm{tr}(\rho_i M_{K+1}) \leq \mathrm{tr}(\Lambda M_{K+1}) = 0 \quad (\therefore =)$$

Intuitively, the measurement has an error if the state is ρ_i but the outcome is $j \neq i$.

How good is the "pretty good" meas?

- If ρ_i's are orthogonal, it is perfect.
- Upon "googling" many hits on PGM being optimal in specific applications.
- For pure states $\rho_i = |\psi_i\rangle\langle\psi_i|$ given equiprobably, ave prob error $\leq \frac{1}{K} \sum_{i \neq j} |\langle\psi_i|\psi_j\rangle|^2$

We'll use packing lemma to bdd error prob.

② Gentle measurement lemma (Winter ...):

Let $\rho \geq 0$, $\mathrm{tr}(\rho) \leq 1$, $0 \leq E \leq I$

If $\mathrm{Tr}(\rho M) \geq 1 - \varepsilon$

then $\exists U$ s.t. $\| \underbrace{U E^{\frac{1}{2}} \rho E^{\frac{1}{2}} U^{\dagger}}_{\text{best possible post meas state for the outcome corr to } E} - \rho \|_{tr} \leq \sqrt{8\varepsilon}$

Interpretations:

Given a state ρ, if meas yields 1 outcome whp then the state is hardly changed by the meas.

(3) Def: for a set of states $S = \{\rho_1, \rho_2, \ldots \rho_k\}$

let distinguishability error of S be

$de(S) = \min_{\text{meas}} \text{ave}_i \Pr(\text{outcome} = j \mid \text{state} = \rho_i)$

NB Can upper bound $de(S)$ by considering specific measurements.

(4) The packing lemma:

Let $p_x = \text{prob}(x)$, ξ_x states, $\xi = \sum_x p_x \xi_x$

If projectors Π, Π_x exist s.t. $\forall x$:

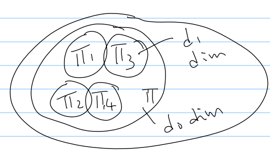
1. $\text{Tr}(\xi_x \Pi) \geq 1-\varepsilon$
2. $\text{Tr}(\xi_x \Pi_x) \geq 1-\varepsilon$
3. $\text{Tr}(\Pi_x) \leq d_1$
4. $\Pi \xi \Pi \leq \frac{\Pi}{d_0}$

☆ (5) $S = \{\gamma_1, \ldots \gamma_K\}$, each $\gamma_x = \xi_x$ wp p_x (iid), $K = \frac{d_0}{d_1} f$ ← +ve fraction

Then $\mathbb{E}_S\, de(S) \leq 2(\varepsilon + \sqrt{8\varepsilon}) + 4f$ (achieved w/ "PGM").

What does this lemma mean?

Conditions (2) & (3) say each ξ_x lives in some d_1-dim space (up to ε approx) defined by ξ_x. Condition (3) says all ξ_x lives in a space defined by Π.



Condition (4) says $\Pi \xi \Pi \leq \frac{\Pi}{d_0}$

Since $\text{tr}(\xi_x \Pi) \geq 1-\varepsilon$, $\Pi \xi_x \Pi \approx \xi_x$

& $\Pi \xi \Pi \approx \xi$

$\therefore$ Condition (4) bounds the max eigenvalue of ξ to be no more than $\frac{1}{d_0}$, or $d_0 \approx \frac{1}{\lambda_{max}(\xi)}$

where $\lambda_{max}(\Gamma)$ denotes the max eigenvalue of Γ.

In general: for 2 rank pure states $|\psi\rangle$ & $|\phi\rangle$

$\lambda_{max}(|\psi\rangle\langle\psi| + |\phi\rangle\langle\phi|)$ is large when $|\psi\rangle$ & $|\phi\rangle$ have high overlap.

In the current problem: $\xi = \sum_x p_x \xi_x$

We expect $\lambda_{max}(\xi)$ to be small if the ξ_x's are distinguishable but having mixed state ξ_x complicates things.

eg $\xi_1 = \left(\frac{1}{4}|0\rangle\langle 0| + \ldots\right)$

$\xi_2 = \left(\frac{1}{4}|0\rangle\langle 0| + \ldots\right)$

Here .... are smaller terms that do not enter the calculation of $\lambda_{max}(\xi_1 + \xi_2)$

None the less, ... still affect their distinguishability.

This problem is worse if $\text{rank}(\xi_{1,2})$ are large.

The packing lemma tells us, if we're communicating using quantum states ρ_x and we know little about them except each lives under Π_x of dim d_1, then we can send $k = \frac{d_0}{d_1} f$ message w/o much error

f — the fraction of space we're willing to leave blank

$\frac{d_0}{d_1}$ — this comes from how distinguishable ρ_x's are.

(Also $d_0 \geq \text{rank}(\Pi)$, $d_1 \approx$ size of ρ_x $\therefore \frac{d_0}{d_1}$ sounds right.)

eg. $|\psi_0\rangle = \sqrt{1-p}\,|0\rangle + \sqrt{p}\,|1\rangle$, $\rho_0 = |\psi_0\rangle\langle\psi_0|$, $p_0 = \frac{1}{2}$

$|\psi_1\rangle = \sqrt{1-p}\,|0\rangle - \sqrt{p}\,|1\rangle$, $\rho_1 = |\psi_1\rangle\langle\psi_1|$, $p_1 = \frac{1}{2}$

$\rho = \begin{bmatrix} 1-p & 0 \\ 0 & p \end{bmatrix}$

Choose $\Pi_0 = |\psi_0\rangle\langle\psi_0|$, $\Pi_1 = |\psi_1\rangle\langle\psi_1|$ $\therefore d_1 = 1$

If we choose $\Pi = |0\rangle\langle 0|$ (when p large)

$\Pi \rho \Pi \leq \frac{\Pi}{d_0}$, $d_0 = 1$ Then $k = 1$

If we choose $\Pi = I$ (when $p \doteq \frac{1}{2}$)

$\Pi \rho \Pi \leq \frac{\Pi}{d_0}$ for $d_0 = 2$ Then $k = 2$

Proof: let $\gamma_i = \rho_{x_i}$. Define a "PGM" using $\Pi \Pi_{x_i} \Pi$ & lower the expected prob of error.

$\Lambda_i = \Pi \Pi_{x_i} \Pi$, $\Lambda = \sum_{i=1}^{K} \Lambda_i$,

$M_i = \Lambda^{-\frac{1}{2}} \Lambda_i \Lambda^{-\frac{1}{2}}$, $M_{K+1} = I - \sum_{i=1}^{K} M_i$

$d_e(S) \leq \frac{1}{K} \sum_i \text{Tr}\, \gamma_i (I - M_i)$

The $\Lambda^{-\frac{1}{2}}$ in M_i is nasty....

Use $I - M_i = I - \Lambda^{-\frac{1}{2}} \Lambda_i \Lambda^{-\frac{1}{2}} = I - \left(\Lambda_i + \sum_{j\neq i} \Lambda_j\right)^{-\frac{1}{2}} \Lambda_i \left(\Lambda_i + \sum_{j \neq i} \Lambda_j\right)^{-\frac{1}{2}}$

$\boxed{\begin{array}{l} \text{op ineq:} \\ I - (X+Y)^{-\frac{1}{2}} X (X+Y)^{-\frac{1}{2}} \\ \leq 2(I-X) + 4Y \end{array}}$

$\leq 2\underbrace{(I - \Lambda_i)} + 4\underbrace{\sum_{j \neq i} \Lambda_j}$

sensible approxs to $I - M_i$ if $\Lambda^{-\frac{1}{2}} \approx I$

$\boxed{\gamma_i = \rho_{x_i} \text{ - r.v}}$

Since $\gamma_i \geq 0$,

$d_e(S) \leq \frac{1}{K} \sum_i \text{Tr}\, \gamma_i (I - M_i)$

$\leq \frac{2}{K} \sum_i \underbrace{\text{Tr}\, \gamma_i (I - \Lambda_i)}_{} + \frac{4}{K} \sum_i \left[\sum_{j \neq i} \text{Tr}(\gamma_i \Lambda_j)\right]$

Next page

$1 - \text{Tr}(\rho_{x_i} \Pi \Pi_{x_i} \Pi)$

$= 1 - \text{Tr}(\Pi \rho_{x_i} \Pi \cdot \Pi_{x_i})$

$\boxed{\begin{array}{l} \text{if } \|X-Y\|_{tr} \leq c \\ \text{then } \forall\, 0 \leq P \leq I \\ |\text{tr}\, P(X-Y)| \leq c \end{array}}$ + gentle meas + 1

$\leq 1 - \text{Tr}(\rho_{x_i} \cdot \Pi_{x_i}) + \sqrt{8\varepsilon}$

$\leq \varepsilon + \sqrt{8\varepsilon}$ (indep of i $\therefore \frac{1}{K}\sum_i$ cancel out)

by 2

→ from previous page, seeking a bound for $\sum_j \text{Tr}(\gamma_i \Lambda_j)$ for $j \neq i$

Here use the fact $\gamma_i = \rho_x$ wp x, drawn iid.

$\underset{p_1 p_2 \cdots p_K}{\mathbb{E}} \sum_{j\,(j \neq i)} \text{Tr}(\gamma_i \Lambda_j) = \sum_{j\,(j\neq i)} \text{Tr}\left[\left(\mathbb{E} \rho_{x_i}\right) \Pi \left(\mathbb{E}\, \Pi_{x_j}\right) \Pi\right]$

independent

$\mathbb{E} \rho_{x_i} = \rho$

$\leq \sum_j \text{Tr}(\Pi \rho \Pi)(\mathbb{E}\, \Pi_{x_j})$

3 $\leq \sum_j \text{Tr}\left(\frac{\Pi}{d_0}\right)\left(\mathbb{E}\, \Pi_{x_j}\right)$ 4

$= \mathbb{E} \sum_j \text{Tr}\left(\frac{\Pi}{d_0}\right) \Pi_{x_j}$ (rank $\leq d_1$ eigenvalues $= 0, 1$)

$\leq \mathbb{E} \sum_j \frac{d_1}{d_0} = K \frac{d_1}{d_0} \leq f$.

Back to earlier Qn:

Let $\boxed{Q}$ be a box that emits ρ_x to Bob if Alice inputs x.

For any n, consider an (M, n) code transmitting $\log M$ bits to Bob by n uses of Q.

Let P_e = worst prob error, $\mathbb{E} P_e$ expected prob error

R achievable if there are $(2^{nR}, n)$ codes w/ $P_e \to 0$

Call capacity of Q, $C(Q) = \sup R$ achievable.

Theorem: $C(Q) = \max_{p_x} \chi(\{p_x, \rho_x\})$

Again, need a direct coding proof & a converse proof.

Converse:

Consider the state $\sum_{x_1 x_2 \cdots x_n} \overbrace{p(x_1 x_2 \cdots x_n)}^{\text{arbitrary}} |x_1 \cdots x_n\rangle\langle x_1 \cdots x_n| \otimes \rho_{x_1} \cdots \otimes \rho_{x_n}$

system labels $X_1 X_2 \cdots X_n$ $B_1 \cdots B_n$

$nR \leq I(X_1 \cdots X_n : B_1 \cdots B_n)$ ← Holevo info n-shot arbitrary probs

$= S(B_1 \cdots B_n) - S(B_1 \cdots B_n | X_1 \cdots X_n)$

$\leq \sum_{i=1}^{n} S(B_i) - \sum_{x_1 \cdots x_n} p(x_1 \cdots x_n) S(\rho_{x_1} \otimes \cdots \otimes \rho_{x_n})$

Need $B_1 \cdots B_n$ in product state $= \sum_{i=1}^{n} S(B_i) - \sum_{x_1 \cdots x_n} p(x_1 \cdots x_n) \sum_{i=1}^{n} S(\rho_{x_i})$

$= \sum_{i=1}^{n} S(B_i) - \sum_{i=1}^{n} \underbrace{p(x_i)}_{\text{marginal}} S(\rho_{x_i}) \leq n\, I(X_i : B_i)$

arbitrary probs 1-shot Holevo info

Direct coding:

Recall in Shannon's noisy coding theorem

$C_1 = x_{11}\, x_{12} \cdots x_{1n}$
$C_2 = x_{21}\, x_{22} \cdots x_{2n}$
$\vdots$
$C_M = x_{M1}\, x_{M2} \cdots x_{Mn}$

$x_{ij} \sim p_x$ iid
whp these are typical sequences
(prob of outcome $\approx 2^{-nH(X)}$)

Here, we demand C_i's be drawn randomly among strongly typical sequences (next page) and to send message i Alice inputs $x_{i1}, x_{i2} \cdots x_{in}$ into the n uses of $\boxed{Q}$.

$\therefore$ States to be distinguished by Bob:

$\gamma_i = \rho_{x_{i1}} \otimes \rho_{x_{i2}} \otimes \cdots \otimes \rho_{x_{in}}$

Strongly typical sequence:

For r.v X, a sequence $C = x_1 x_2 \cdots x_n$ is ε-strongly typical if the empirical dist^n of X observed in C is ε-close to $\{p_x\}$

Say for $x=1$, $q(1) = \frac{1}{n}(\#\, x_i\text{'s equal to } 1)$

for $x=2$, $q(2) = \frac{1}{n}(\#\, x_i\text{'s} = 2)$

$\vdots$

and $\| q - p \|_1 = \sum_x |q(x) - p(x)| \leq \varepsilon$

HW: ε-strongly typical sequences are ε' typical

---

eg. let $\Omega = \{1,2,3,4\}$, $p(a) = \frac{a}{10}$, $n = 20$

$C = 3\,3\,3\,4\,4\;\; 2\,1\,3\,2\,2\;\; 2\,4\,3\,4\,3\;\; 4\,2\,4\,4\,3$

$q(1) = 1/20$	$p(1) = 1/10$
$q(2) = 5/20$	$p(2) = 2/10$
$q(3) = 7/20$	$p(3) = 3/10$
$q(4) = 7/20$	$p(4) = 4/10$

$\|p - q\|_1 = 0.2$. So C is 0.2-strongly typical.

NB we focus on $n \gg |\Omega|$

Given $C = x_1 \cdots x_n$ ε-strongly typical

What do we know about $\gamma_c = \rho_{x_1} \otimes \cdots \otimes \rho_{x_n}$?

Up to reordering the n systems, it's $\bigotimes_x \rho_x^{\otimes n q(x)}$ ← empirical prob

For each x, by discussion leading to quantum data compression, there's a projector Π_x s.t.

$\mathrm{Tr}(\Pi_x \rho_x^{\otimes n q(x)}) \geq 1 - \delta_1$

$\dim \Pi_x \leq 2^{-n q(x)(S(\rho_x) + \varepsilon_1)}$

$\therefore$ Given C, let $\Pi_C = \bigotimes_x \Pi_x$ ← acting on the correct systems

$\mathrm{tr}[\Pi_C\, \rho_{x_1} \otimes \cdots \otimes \rho_{x_n}] \geq 1 - \delta_2$

$\dim \Pi_C \leq 2^{-n[\sum_x q(x) S(\rho_x) + \varepsilon_2]}$

eg. when $C = 33344\ 2132224343\ 42443$.

7 systems compressed by Π_4

state $\gamma_C = \underbrace{\rho_3 \otimes \rho_3 \otimes \rho_3 \otimes \overbrace{\rho_4 \otimes \rho_4 \otimes \cdots \otimes \rho_4}^{} \otimes \rho_3}_{\text{compressed by } \Pi_3}$

for $x=1$, Π_1 acts on sys 7

$x=2$, Π_2 acts on sys 6, 9, 10, 11, 17

$x=3$, Π_3 acts on sys 1, 2, 3, 8, 13, 15, 20

$x=4$, Π_4 acts on sys 4, 5, 12, 14, 16, 18, 19

More precisely,

for each n, for c, γ, Π_c defined earlier,

& let $\rho = \sum_x p_x \rho_x$, Π projector onto typical space of $\rho^{\otimes n}$

$\exists\ \varepsilon_n, \delta_n$ s.t.

① $\mathrm{Tr}(\gamma_c \Pi_c) \geq 1 - \delta_n$

② $\mathrm{Tr}(\Pi_c) \leq 2^{n[\sum_x p_x S(\rho_x) + \varepsilon_n]}$

proof ex — ③ $\mathrm{Tr}(\gamma_c \Pi) \geq 1 - \delta_n$

④ $\Pi\, \rho^{\otimes n}\, \Pi \leq 2^{-n[S(\rho) - \varepsilon_n]}\ \Pi$

proof outlined already

(and $\varepsilon_n, \delta_n \to 0$ as $n \to \infty$)

Finally, back to direct coding proof for $C(Q)$:

The M messages are encoded as: $\gamma_1, \gamma_2, \ldots, \gamma_M$

Where $\gamma_i = \rho_{x_{i1}} \otimes \rho_{x_{i2}} \otimes \cdots \otimes \rho_{x_{in}} =: \rho_{C_i}$

for $C_i = x_{i1} x_{i2} \cdots x_{in}$ a randomly chosen strongly typical sequence

The hacking lemma now applies

ξ_x being ρ_c, c strongly typical

Π_x being Π_c

d_1 ...... $2^{n[\sum_x p_x S(\rho_x) + \varepsilon_n]}$

Π ...... Π

d_0 ...... $2^{n[S(\sum_x p_x \rho_x) + \varepsilon_n]}$

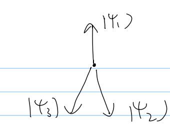
K ...... M

$\therefore M$ can be $\frac{d_0}{d_1} \cdot f$

$\approx 2^{n[\chi(\{p_x, \rho_x\}) - \delta_3]}$

takes care of f & ε_n

eg. Consider the states



$\rho_1 = |\psi_1\rangle\langle\psi_1|\ 0.9 + 0.1\ \frac{I}{2}$ for $|\psi_1\rangle = |0\rangle$

$\rho_2 = |\psi_2\rangle\langle\psi_2|\ 0.9 + 0.1\ \frac{I}{2}$ for $|\psi_2\rangle = \cos\theta\, |0\rangle + \sin\theta\, |1\rangle$

$\rho_3 = |\psi_3\rangle\langle\psi_3|\ 0.9 + 0.1\ \frac{I}{2}$ for $|\psi_3\rangle = \cos\theta\, |0\rangle - \sin\theta\, |1\rangle$

s.t. $0.4\rho_1 + 0.3\rho_2 + 0.3\rho_3 = \frac{I}{2}$

So, the Q box emits one of $\rho_{1,2,3}$ on demand.

Applying what we proved, $C(Q) = \max_{p_1 p_2 p_3} \chi(\{p_i, \rho_i\})$

Since $S(\rho_i)$ same, $\chi(\{p_i, \rho_i\}) = S(\sum_i p_i \rho_i) - H(\{0.05\})$.

max by $\sum_i p_i \rho_i = \frac{I}{2}$, or $p_1 = 0.4$, $p_2 = p_3 = 0.3$.

$=1$ $\qquad = 0.29$

What should Alice send to Bob?

Consider all strings of 1 2 3's of length n

with $\approx 40\%$ 1's, 30% 2's, 30% 3's.

$H(\{0.4, 0.3, 0.3\}) \doteq 1.571$

There are $\approx 2^{n(1.571+\varepsilon)} \approx 2.971^n$ such strings.

Alice is drawing $2^{n(0.71-\delta)}$ strings from this pool (with replacement). If $\delta = 0.02$, there are $120 = M$ messages.

Say $C_1 = 2\,1\,1\,2\,3\,1\,2\,1\,3\,3$

$C_2 = 1\,3\,2\,1\,3\,2\,1\,2\,1\,3$

$\vdots$

$C_{120} = 2\,1\,3\,1\,2\,3\,1\,1\,2\,3$

$\boxed{\text{n = 10 for illustration. Side Qn: lower bound Iacc for n = 10}}$

So $\gamma_1 = \rho_2 \rho_1 \rho_1 \rho_2 \rho_3 \rho_1 \rho_2 \rho_1 \rho_3 \rho_3$

$\gamma_2 = \rho_1 \rho_3 \rho_2 \rho_1 \rho_3 \rho_2 \rho_1 \rho_2 \rho_1 \rho_3$

$\vdots$

$\gamma_{120} = \rho_2 \rho_1 \rho_3 \rho_1 \rho_2 \rho_3 \rho_1 \rho_1 \rho_2 \rho_3$

product states, but now classically correlated once $C_1, C_2 \ldots C_{120}$ fixed.

Bob's measurement: $\boxed{\Lambda_i = \Pi\,\Pi_{X_i}\Pi\ , \quad \Lambda = \sum_{i=1}^{K} \Lambda_i\ , \quad M_i = \Lambda^{-\frac{1}{2}}\Lambda_i\Lambda^{-\frac{1}{2}}\ , \quad M_{K+1} = I - \sum_{i=1}^{K} M_i}$

e.g. $\gamma_{1000} = \rho_2\rho_1\rho_3\rho_1\rho_2\rho_3\rho_1\rho_1\rho_2\rho_3$

Compress these 4 ($S(\rho_1) = H(0.05) \doteq 0.29$)

down to $2^{\overbrace{4(\underset{S(\rho_1)}{0.29} + \underset{\varepsilon}{0.1})}} \simeq 3$ dim

Compress to $2^{3(0.29+0.1)} \simeq$ "2.25" dims

∴ $\Pi_{120} = \otimes$ of the 3 projections

Tr or rank of $\Lambda_{120} \approx$ "2.25"$^2 \times 3 \approx 14.5$

≈ 15 dims $\ll d_1$

Theoretically, should be $2^{10(0.29+0.1)} \approx 15$ also.

Π projection onto typical space of $\left(\sum_i p_i \rho_i\right)^{\otimes 10}$

which is the full space. $d_0 \approx 2^{10}$

"Can send" $2^{10(\underbrace{1-0.29}_{\chi}-0.2)} \approx 120$ messages.

So $\Pi\,\Pi_{120}\,\Pi \simeq \Pi_{120}$.

Same for $\Pi_1 \cdots \Pi_{119}$.

$\Lambda = \sum_{i=1}^{120} \Pi_i$ ← highly collective

∴ $\{M_i = \Lambda^{-\frac{1}{2}}\,\Pi_i\,\Lambda^{-\frac{1}{2}}\}$ defines a highly collective measurement

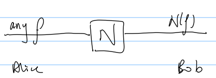
What is the I_{acc} of the ensemble $\mathcal{E} = \{p_i, \rho_i\}$ in the above example?

That of the trine state $\simeq 0.585$

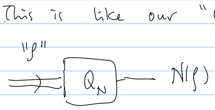
I_{acc} for the current ensemble should be even lower.

Now, classical capacity of a quantum channel.

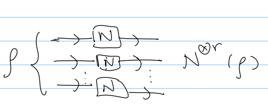
Basic use:



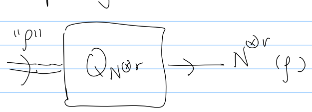
This is like our "Q" box:



Alice can also use $N^{\otimes r}$:



Corresponding Q box:



It follows from the capacity discussion of the Q box that

$$C(N) \geq \sup_r \frac{1}{r} \underbrace{\chi(N^{\otimes r})}$$

and $\leq$

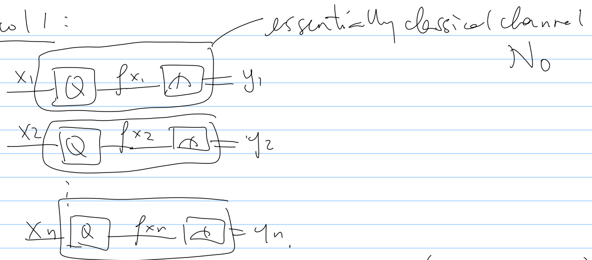
defined as

$$\max_{\{p_x,\rho_x\}} \chi(\{p_x, N^{\otimes r}(\rho_x)\})$$

↑ input to n channels.

This is called the HSW theorem, after Holevo, Schumacher & Westmoreland, PRA 56, 131 (1997), IEEE TIT 44, 269 (1998).

Protocol 1:

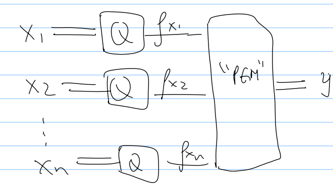


Alice chooses $p(x)$ to max $I_{acc}(\{p(x), \rho_x\})$

(The comm capacity of N_0)

Protocol: Shannon's direct coding method for N_0

Protocol 2:

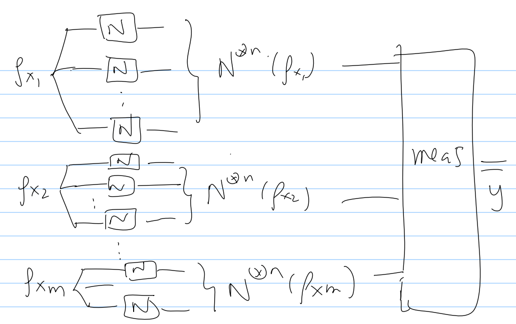


Alice chooses $p(x)$ to max $\chi(\{p(x), \rho_x\})$

and this is the capacity $C(Q)$

Protocol: codewords randomly chosen from strongly typical sequences.

Protocol 3:



Max over $\{p_x, \rho_x\}$ ← n use input, n use output

$$\text{Capacity} = C(N) = \sup_n \frac{1}{n} \chi(p_x, N^{\otimes n}(\rho_x))$$

Note that the capacity expression has an optimization over n — called a "regularized" expression in contrast to the capacity expression for the classical channel (or the Q box which involve only 1 copy of the resource (these are called single letterized expression).

The step $S(B_1 \cdots B_n) = S(B_1) + S(B_2) + \cdots + S(B_n)$ in the converse of capacity of Q-box breaks down when output state $N^{\otimes n}(\rho)$ is entangled in the channel setting.

To do:

- Some examples | problems like not having upper bounds to capacity of AD channel
- Additivity results vs non additivity (this really says where the problem come from)
- brief discussion on the nonadditivity example