Theorem [Shannon's noisy coding theorem]
$C(N) = \max_{p(x)} I(X:Y)$

How to prove this?

1. Direct coding – consider codes that are promising

   A clever code doesn't come by easily.
   Instead, consider "random" (M,n) codes with rate = $I(X:Y)$ and show Prob($EP_e \to 0$) > 0.

   Thus $\exists$ code with small EP_e (our 2nd encounter with "existential proofs"). Extract a subcode with similar rate but $P_e \to 0$.

2. Converse – show that if at higher rates, $EP_e \not\to 0$.

Plan: 2, then heuristic 1, then 1.

Proof of converse:

$$nR = H(M) \overset{\text{checking ...}}{=} H(M|Y^n) + I(M:Y^n) \le H(M|Y^n) + I(E_n(M):Y^n)$$

(i) data processing ineq

small if $EP_e \to 0$
(ii) $H(M|Y^n) \le 1 + P_e\, nR$

(iii) by lemma $\le n \max_{p(x)} I(X:Y)$

(i) Data processing inequality $I(E:F) \ge I(E:G)$
if $E \to F \to G$ is a Markov Chain
(i.e. $I(E:G|F) = 0$)

Proof:

$$\begin{aligned} I(E:FG) &= H(E) + H(FG) - H(EFG) \\ &= I(E:G) + H(E|G) + H(FG) - H(EFG) \\ &= I(E:G) + H(E|G) + - H(E|FG) \\ &= I(E:G) + I(E:F|G) \end{aligned}$$

but the LHS is symmetric wrt exchange F and G, so must the RHS.

So, $I(E:G) + \underbrace{I(E:F|G)}_{\ge 0} = I(E:F) + \underbrace{I(E:G|F)}_{0}$

So, $I(E:G) \ge I(E:F)$.

(ii) Thm [Fanos ineq]:

Let $P_e = \text{prob}(X \neq Z)$, $Z = f(Y)$, Ω=sample space of X.
Then, $H(P_e) + P_e \log(|\Omega|-1) \ge H(X|Y)$

Proof: Define new rv E, E=0 if X=Z, E=1 otherwise.

$$H(EX|Y) = H(X|Y) + H(E|XY)$$
$$H(EX|Y) = H(E|Y) + H(X|EY)$$

$$\begin{aligned} \text{So, } H(X|Y) &= H(E|Y) + H(X|EY) \\ &\le H(E) + \textstyle\sum_y p(y)\, [P_e H(X|E=1\, Y=y) + (1-P_e) H(X|E=0\, Y=y)] \\ &\le H(P_e) + P_e \log(|\Omega|-1) \end{aligned}$$

Making the replacements:
$M \leftrightarrow X$
$Y^n \leftrightarrow Y$
$2^{nR} \leftrightarrow |\Omega|$ gives $H(M|Y^n) \le 1 + P_e\, nR$

(iii) Lemma: Let $Y^n = N^{\otimes n}(X^n)$.
Then, $I(X^n:Y^n) \le \sum_{i=1}^n I(X_i:Y_i)$.

Pf: $I(X^n:Y^n) = H(Y^n) - H(Y^n|X^n)$

$$\begin{aligned} &= H(Y^n) - \textstyle\sum_{i=1}^n H(Y_i|Y_1 \ldots Y_{i-1} X^n) && \text{Chain rule} \\ &= H(Y^n) - \textstyle\sum_{i=1}^n H(Y_i|X_i) && Y_i \text{ only depends on } X_i \\ &\le \textstyle\sum_{i=1}^n H(Y_i) - \sum_{i=1}^n H(Y_i|X_i) && \text{Subadditivity} \\ &\le \textstyle\sum_{i=1}^n I(X_i:Y_i) \end{aligned}$$

1. Direct coding:

Let $M = 2^{n(I(X:Y)-\delta n)}$. What's the (M,n) code?

Fix any p(x).

Encoder $\mathcal{E}_n$:
Pick M codewords $c_i = x_{i1} \ldots x_{in}$
each x_{ij} chosen iid $\sim p(x)$).

Fixed & known to Alice & Bob once choosen.

$c_1 = x_{11}, x_{12}, \ldots, x_{1n}$
$c_2 = x_{21}, x_{22}, \ldots, x_{2n}$
.
.
$c_M = x_{M1}, x_{M2}, \ldots, x_{Mn}$

Everything refers to this particular code $\mathcal{C}_n$ from now on.

1. Direct coding: $c_1 = x_{11}, x_{12}, \ldots, x_{1n}$
$c_2 = x_{21}, x_{22}, \ldots, x_{2n}$ x_{ij} chosen iid $\sim p(x)$
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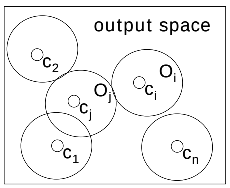
Heuristically why $P_e \to 0$:

The n channel outputs Y^n is iid with $p(y) = \sum_x p(y|x)\, p(x)$

With high prob, output typical $y_1 \cdots y_n$, $\approx 2^{nH(Y)}$ of them.



For each c_i sent via $N^{\otimes n}$, there're $\approx 2^{nH(Y|X)}$ possible outcomes (call the set O_i) centered around c_i .

Since the c_i's are random, if $2^{nH(Y|X)}\, M << 2^{nH(Y)}$, these O_i's don't overlap much. So, decoder just output "which sphere" contains the output $y_1 \cdots y_n$.

1. Direct coding: $c_1 = x_{11}, x_{12}, \ldots, x_{1n}$
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Better proof why $P_e \to 0$.

Long version (18 pages) available in homepage, here, shrink to 6 pages, skipping most detail, esp ε, δ ignored.

Recall:

Def[typical sequence]:

x^n ε-typical if $|-1/n \log(p(x^n)) - H(X)| \leq \varepsilon$

It means $2^{-n(H(X)+\varepsilon)} \leq p(x^n) \leq 2^{-n(H(X)-\varepsilon)}$.

Def[Jointly typical sequence]:

$x^n y^n$ ε-jointly-typical if

$$|-1/n \log(p(x^n y^n)) - H(XY)| \leq \varepsilon$$

where $p(x^n y^n) = \Pi_{i=1}^n\, p(x_i\, y_i)$.

Need also: (a) $|-1/n \log(p(x^n)) - H(X)| \leq \varepsilon$
(b) $|-1/n \log(p(y^n)) - H(Y)| \leq \varepsilon$

[The strong typicality has (c) $\Rightarrow$ (a,b), but not for entropic typicality.]

Def[Jointly-typical set]: $A_{n,\varepsilon} = \{x^n y^n\ \varepsilon\text{-jointly typical}\}$

<u>Joint asymptotic equipartition (Joint AEP) theorem:</u>

Let (X^n, Y^n) be sequences of length n
drawn iid according to $p(x^n\, y^n) = \Pi_{i=1}^n\, p(x_i\, y_i)$.

Then:

1. $\Pr(X^n Y^n \in A_{n,\varepsilon}) \to 1$
2. $|A_{n,\varepsilon}| \approx 2^{nH(XY)}$
3. if we draw X^n & Y^n according to $q(x^n\, y^n) = p(x^n)\, p(y^n)$.
   $\Pr_q$ (outcome $\in A_{n,\varepsilon}$) $\approx 2^{-nI(X:Y)}$

Proof (with ε, δ) available in the 18 page notes.

More observations:

Given $y^n \in T^Y_{n,\varepsilon}$, how many $x^n \in T^X_{n,\varepsilon}$ is s.t. $x^n\, y^n \in A_{n,\varepsilon}$?

Call this set $S(y^n)$.

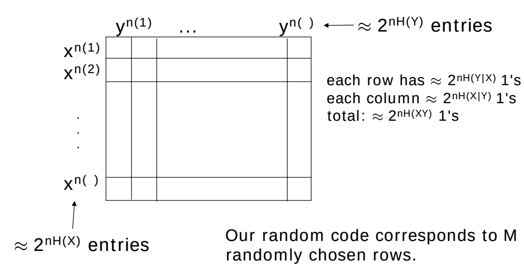
(1) $p(x^n|y^n) = p(x^n y^n) \,/\, p(y^n) \approx 2^{-n[H(XY)-H(Y)]} = 2^{-n[H(X|Y)]}$
$\uparrow$ since $x^n y^n \in A_{n,\varepsilon}$!!

(2) $1 = \sum_{x^n \in S}\, p(x^n|y^n) \approx |S(y^n)|\ 2^{-n[H(X|Y)]}$

Hence, $|S(y^n)| \approx 2^{nH(X|Y)}$. Fraction of such $x^n \approx 2^{-nI(X:Y)}$.

Similarly, given $x^n \in T^X_{n,\varepsilon}$, $\approx 2^{nH(Y|X)}$ y^n's are jointly typical with it, and the fraction of such $y^n \approx 2^{-nI(X:Y)}$.

Make a table of typical x^n's and y^n's, and for jointly typical $x^n y^n$, put a 1, else, put a 0.



Our random code corresponds to M randomly chosen rows.

D_n: typical set decoding

Given y^n, if there is a unique $x^n \in S(y^n)$,
output m' s.t. $c_m = x^n$.
Else, output W=M+1 (error symbol).

How will this fail for message m?

Either - no such x^n — Err_0 unlikely–only when $x^n y^n$ not jointly typical

- or $\exists m'' \neq m$ with $c_{m''} y^n \in A_{\varepsilon,n}$ — $Err_{m''}$

For the random code $\mathcal{C}_n$, let the average error over all messages be $EP_e(\mathcal{C}_n)$, same as error if m=1 (since all messages similar).

$$EP_e(\mathcal{C}_n) = Pr_{\mathcal{C}_n}(W \neq 1 | m=1) = Pr_{\mathcal{C}_n}(Err_0 \cup \underbrace{Err_2 \ldots \cup Err_M}_{\text{equiprobable}} | m=1)$$

union bdd $\leq M \, Pr_{\mathcal{C}_n}(Err_2 | m=1)$

Bounding $Pr_{\mathcal{C}_n}(Err_2 | m=1) = Pr_{\mathcal{C}_n}(c_2 y^n \in A_{n,\varepsilon_n})$:

But c_2 and $y^n = N^{\otimes n}(x_1)$ independent.

By joint AEP [3], $Pr_{\mathcal{C}_n}(c_2 y^n \in A_{n,\varepsilon_n}) \approx 2^{-nI(X:Y)}$

If $M = 2^{n(I(X:Y) - \delta_n)}$ and $n\delta_n$ growing with n but $\delta_n \to 0$

$EP_e(\mathcal{C}_n) \leq M \, Pr_{\mathcal{C}_n}(Err_2 | m=1) \to 0$

Note: $P_e(m=1) + P_e(m=2) + \ldots + P_e(m=M) = M \, EP_e(\mathcal{C}_n)$

Reorder m's so that $P_e(m)$ is increasing.

So, $P_e(m=1) + P_e(m=2) + \ldots + P_e(m=M/2) \leq M/2 \, EP_e(\mathcal{C}_n)$

So, keeping only codewords for m=1, ..., M/2,
worse case prob error $\leq EP_e(\mathcal{C}_n)/2$.