

Superdense coding:  $1 \text{ ebit} + 1 \text{ qbit} \geq 2 \text{ cbits}$

Teleportation:  $1 \text{ ebit} + 2 \text{ cbits} \geq 1 \text{ qbit}$

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Resource inequality:

$$\boxed{\phantom{\text{Resource Inequality}}} \geq \boxed{\phantom{\text{Resource Inequality}}}$$

Resources on the LHS can be used to produce resource on the RHS.

Usually it means we know a protocol  $P$  for doing so.

Will see "concatenating protocols" correspond to algebraic operations of these resource inequalities.

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Can we use less in superdense coding or teleportation?

To show optimality of superdense coding, suppose

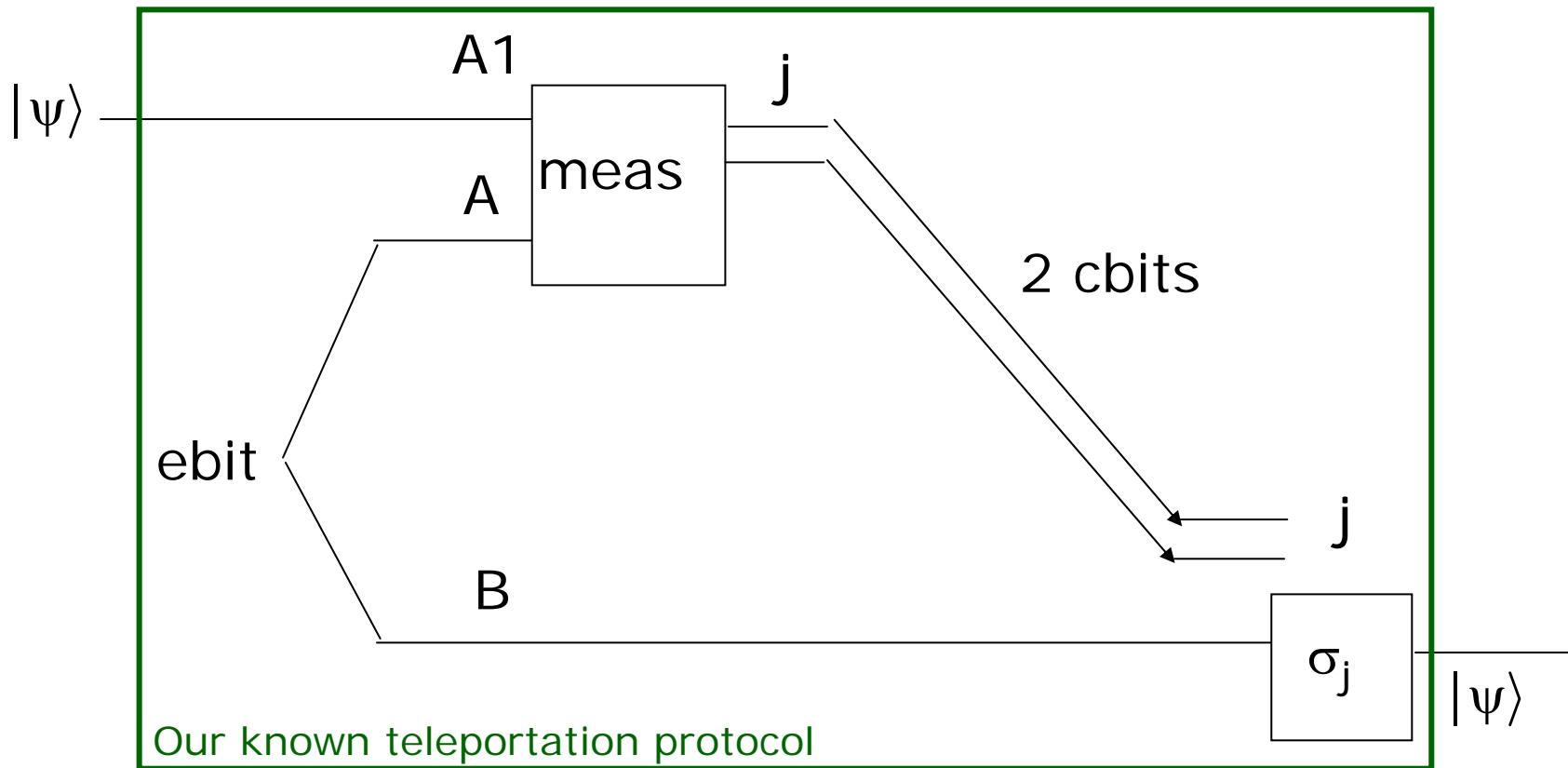
$$\alpha \text{ ebit} + \beta \text{ qbit} \geq 2 \text{ cbits}$$

$\alpha, \beta$  need not be integers if we have an asymptotic (but exact) protocol and we look at the average cost.

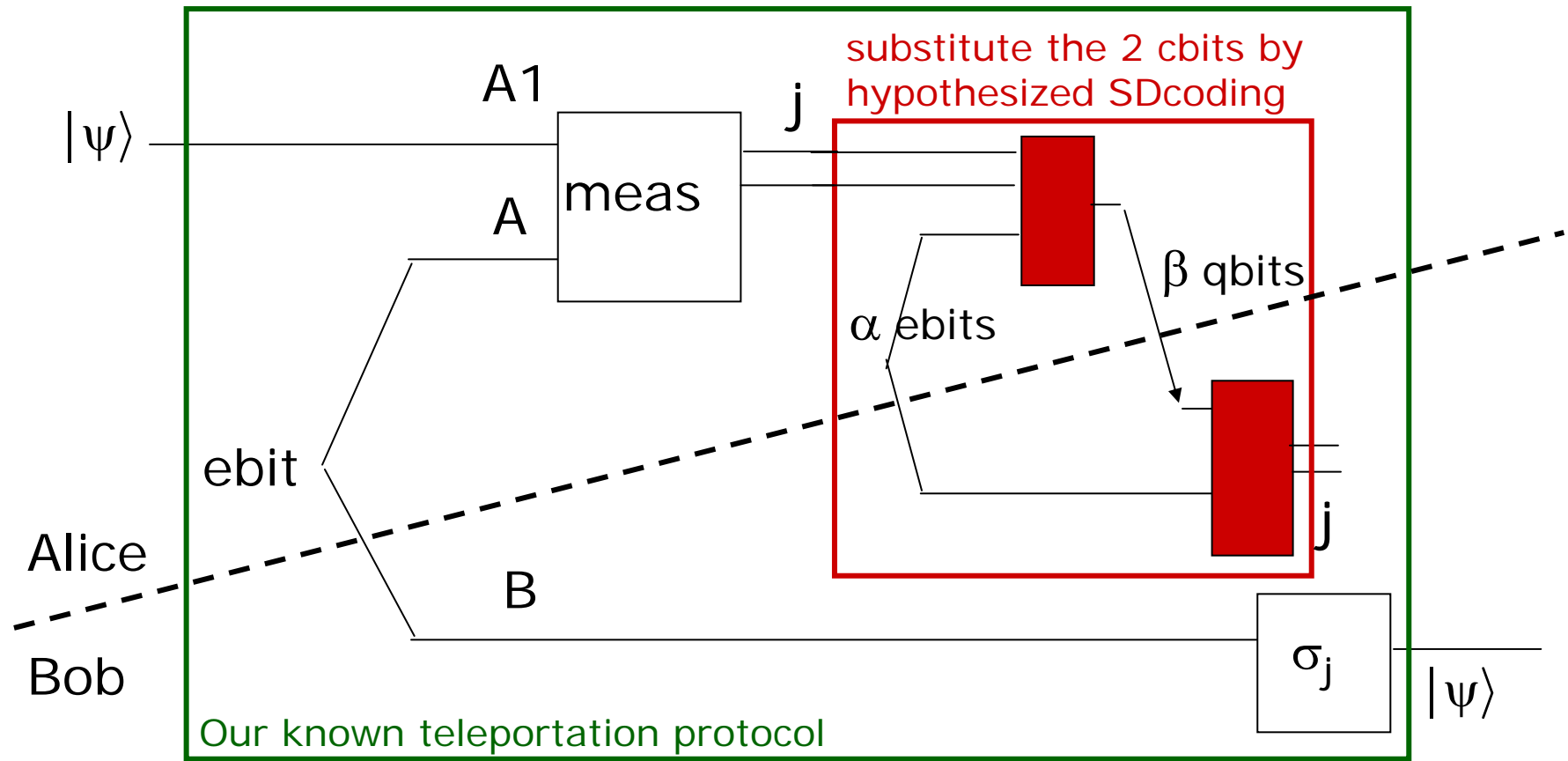
i.e. there is a (hypothesized) protocol to send 2 bits using  $\alpha$  ebit and by sending  $\beta$  qubits (however complicated)

Then, we can use superdense coding to "supply" the 2 cbits needed in the teleportation protocol we already know to be working.

Using the **hypothesized superdense coding** protocol to "supply" the 2 cbits needed in **known teleportation**:



Using the **hypothesized superdense coding** protocol to "supply" the 2 cbits needed in **known teleportation**:



Net: send 1 qubit using  $\beta$  qbit &  $(1 + \alpha)$  ebits. Contradiction if  $\beta < 1$  (for any  $\alpha$ ) (by recursive argument). So,  $\beta \geq 1$ .

Intriguingly, teleportation is the reason why SDcoding is optimal.

Algebraically:

Known to work:

Teleportation: 1 ebit + 2 cbits  $\geq$  1 qbit

For any superdense coding protocol:

$$\alpha \text{ ebit} + \beta \text{ qbit} \geq 2 \text{ cbits}$$

math: substitute the 2nd inequality into the 1st  
operationally: we use the protocol (SD) giving the 2nd inequality as a BB in the protocol giving the 1st inequality. The math is valid if the composition is.

$$\begin{array}{l} 1 \text{ ebit} + (\alpha \text{ ebit} + \beta \text{ qbit}) \geq \\ 1 \text{ ebit} + 2 \text{ cbits} \geq 1 \text{ qbit} \end{array}$$

$$(1 + \alpha) \text{ ebits} + \beta \text{ qbit} \geq 1 \text{ qbit} \quad . \quad \text{So, } \beta \geq 1$$

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For superdense coding, suppose  $\alpha \text{ ebit} + \beta \text{ qbit} \geq 2 \text{ cbits}$

What about the entanglement cost, or  $\alpha$ ?

To analyze this, we use the fact stated last time (to be proved next class) that a quantum state in  $d$  dimensions cannot transmit more than  $d$  classical messages.

Then  $\alpha + \beta \geq 2$  because the LHS is the total number of qubits in Bob's hands in the end.

So, missing ebits have to be made up for by qbits.

Since  $\text{qbit} \geq \text{ebit}$ ,  $\alpha = \beta = 1$  optimal

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Optimality proof for teleportation is similar:

Hypothesize any other means to achieve teleportation:

$$\gamma \text{ ebit} + \delta \text{ cbits} \geq 1 \text{ qbit}$$

To lower bound  $\delta$  (the comm cost of teleportation):

Use any hypothesized teleportation to supply the 1 qbit needed in the known protocol for superdense coding.

Net resource counting for superdensing coding:

send 2 bits using  $\delta$  cbits and  $(1+\gamma)$  ebits

A contradiction unless  $\delta \geq 2$ . (Run recursively to see this).

Alt:  $1 \text{ ebit} + (\gamma \text{ ebit} + \delta \text{ cbits})$   
 $\geq 1 \text{ ebit} + 1 \text{ qbit} \geq 2 \text{ cbits} . \quad \text{So, } \delta \geq 2.$

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Optimality proof for teleportation is similar:

Hypothesize any other means to achieve teleportation:

$$\gamma \text{ ebit} + \delta \text{ cbits} \geq 1 \text{ qbit}$$

To lower bound  $\gamma$  (the entanglement cost of teleportation):

Use teleportation to share an ebit between Alice and Bob.

So  $\gamma$  ebit and  $\delta$  cbits generate 1 ebit.

Contradiction unless  $\gamma \geq 1$ .

Alt:  $\gamma \text{ ebit} + \delta \text{ cbits} \geq 1 \text{ qbit} \geq 1 \text{ ebit}$ , so,  $\gamma \geq 1$ .



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Superdense coding and teleportation invert one another IF entanglement is free. What happens otherwise?

Note that teleportation generates 2 random bits shared between Alice and Bob. Seems like the initial ebit is degraded into these.

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Harrow 03: define cobit (coherent classical communication) as the ability to perform:  $|x\rangle_A \rightarrow |x\rangle_A |x\rangle_B$  for a basis.

c.f.

qbit is the ability to perform  $|x\rangle_A \rightarrow |x\rangle_B$

cbit is the ability to perform  $|x\rangle_A \rightarrow |x\rangle_B |x\rangle_E$

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NB: A cobit is between the ability of a qbit and an ebit:

(1) **qbit**  $\geq$  **cobit**:

Protocol:

Alice applies a unitary  $|x\rangle_A \rightarrow |x\rangle_A |x\rangle_{A1}$

Alice transmits A1 to Bob (**using the qbit**). A1 becomes B.

Net effect:  $|x\rangle_A \rightarrow |x\rangle_A |x\rangle_B$  which **performs a cobit**.

(2) **cobit**  $\geq$  **ebit**

(3) **cobit**  $\geq$  **cbit**

Protocol:

Alice prepares a superposition  $\sum_x |x\rangle_A$

Send A "coherently" **using the cobit**

Net effect: the final state  $\sum_x |x\rangle_A |x\rangle_B$  is **an ebit**.

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Why consider such an odd (if not self-ridiculed) task?

(1) Superdense coding provides more than 2 cbits!  
It leaks no info to the environment. It achieves:

$$1 \text{ ebit} + 1 \text{ qbit} \geq 2 \text{ cobits}$$

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Harrow 03: define cobit (coherent classical communication) as the ability to perform:  $|x\rangle_A \rightarrow |x\rangle_A |x\rangle_B$  for a basis.

(2) Is there an inverse?

$$2 \text{ cobits} \geq 1 \text{ qbit} + 1 \text{ ebit}$$

Not obvious, but let's try:

$$1 \text{ ebit} + 2 \text{ cobits} \geq 1 \text{ qbit} + 2 \text{ ebits} \quad (*)$$

(\*) is suggestive – if there is a protocol giving (\*), it's like doing teleportation using cobits and recycling 2 ebits

## Teleportation:

- Alice (sender) & Bob (receiver) have initial state  $|00\rangle + |11\rangle_{AB}$
- Alice's message is  $|\psi\rangle_{A1} = a|0\rangle + b|1\rangle$

- Alice measures A1 & A along the basis:

$$|y_0\rangle = |00\rangle + |11\rangle \quad |y_1\rangle = |10\rangle + |01\rangle \quad |y_2\rangle = |10\rangle - |01\rangle \quad |y_3\rangle = |00\rangle - |11\rangle$$

Prior to the measurement, the state on A1 A B is

$$(a|0\rangle + b|1\rangle)_{A1} (|00\rangle + |11\rangle)_{AB} = \frac{1}{2} \sum_j |y_j\rangle_{A1 A} (\sigma_j |\psi\rangle)_B$$

(just check by expanding both sides)

- If outcome is "j", postmeasurement state is  $|y_j\rangle_{A1 A} (\sigma_j |\psi\rangle)_B$
- Alice sends the outcome j to Bob (using 2 cbits)
- Bob applies  $\sigma_j^{-1}$  to B which now contains  $|\psi\rangle$  regardless of j.

## Teleportation: WITH COBITS

- Alice (sender) & Bob (receiver) have initial state  $|00\rangle + |11\rangle_{AB}$
- Alice's message is  $|\psi\rangle_{A_1} = a|0\rangle + b|1\rangle$

- Alice ~~measures A1 & A along the basis.~~  $|y_j\rangle$  to  $|j\rangle$  transforms  
 $|y_0\rangle = |00\rangle + |11\rangle$     $|y_1\rangle = |10\rangle + |01\rangle$     $|y_2\rangle = |10\rangle - |01\rangle$     $|y_3\rangle = |00\rangle - |11\rangle$

unitary trsf

Prior to the ~~measurement~~, the state on A1 A B is  
 $(a|0\rangle + b|1\rangle)_{A_1} (|00\rangle + |11\rangle)_{AB} = \frac{1}{2} \sum_j |y_j\rangle_{A_1 A} (\sigma_j |\psi\rangle)_B$

After the trsf, it is  $\frac{1}{2} \sum_j |j\rangle_{A_1 A} (\sigma_j |\psi\rangle)_B$

- ~~If outcome is "j", postmeasurement state is  $|y_j\rangle_{A_1 A} (\sigma_j |\psi\rangle)_B$~~

A1 A to Bob using 2 cobits

- Alice sends ~~the outcome j to Bob (using 2 cbits)~~

resulting state:  $\frac{1}{2} \sum_j |j\rangle_{A_1 A} |j\rangle_{B_1 B_2} (\sigma_j |\psi\rangle)_B$

- Bob applies  $\sigma_i^{-1}$  to B conditioned on B1 B2 being j  
final state:  $\frac{1}{2} \sum_j |j\rangle_{A_1 A} |j\rangle_{B_1 B_2} |\psi\rangle_B$ , giving 1 qbit + 2 ebits

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Teleportation with cobits:

$$1 \text{ ebit} + 2 \text{ cobits} \geq 1 \text{ qbit} + 2 \text{ ebits}$$

If we use these new ebits in later rounds of teleportation, **asymptotically** we get the exact inverse of SDC:

$$2 \text{ cobits} \geq 1 \text{ qbit} + 1 \text{ ebit}$$

Therefore:  $2 \text{ cobits} = 1 \text{ qbit} + 1 \text{ ebit}$

So, the answer to the puzzle is that, SDcoding and "teleportation with cobits" are inverses of one another. SDcoding is much better than we thought, and teleportation, naturally defined, has poorer input & output.



## Ensembles:

Def: Let  $X$  be a classical random var with distribution  $q(x)$ .

$E = \{q(x), |\psi_x\rangle\}$  is called an *ensemble of quantum states*.

The classical rv  $X$  induces another rv which is a quantum state.

Interpretation: with prob  $q(x)$ , quantum state is  $|\psi_x\rangle$ .

Let  $\rho = \sum_x q(x) |\psi_x\rangle\langle\psi_x|$  be the "average" state.

e.g., let  $q(1) = 1/2$ ,  $|\psi_1\rangle = |0\rangle$

let  $q(2) = 1/4$ ,  $|\psi_2\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$

let  $q(3) = 1/4$ ,  $|\psi_3\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$

$$\rho = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{4} \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{1}{4} \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 3/4 & 0 \\ 0 & 1/4 \end{pmatrix}$$

$$H(X) = -(\frac{1}{2} \log \frac{1}{2} + \frac{1}{4} \log \frac{1}{4} + \frac{1}{4} \log \frac{1}{4}) = 1.5$$

## Ensembles:

The average state:  $\rho = \sum_x q(x) |\psi_x\rangle\langle\psi_x|$  is hermitian and trace 1, therefore it can be written in terms of its eigenvalues and eigenvectors (spectral decomposition):

$$\rho = \sum_v p(v) |e_v\rangle\langle e_v|$$

von Neumann entropy:

$$S(\rho) = [\text{entropy of eigenvalues of } \rho] = -\text{tr } \rho \log \rho$$

For the example ensemble:

$$\rho = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{4} \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{1}{4} \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 3/4 & 0 \\ 0 & 1/4 \end{pmatrix}$$

$$\text{So, } |e_1\rangle = |0\rangle, |e_2\rangle = |1\rangle, p(0) = 3/4, p(1) = 1/4$$

$$S(\rho) = - 3/4 \log 3/4 - 1/4 \log 1/4 = 2 - \log 3 \approx 0.8113$$

$$\text{but } H(X) = 1.5$$

## Quantum data compression (statement of result):

Suppose we make  $n$  iid draws according to  $E = \{q(x), |\psi_x\rangle\}$ .

i.e. with prob  $q(x_1)q(x_2)\cdots q(x_n)$ , state is  $|\psi_{x_1}\rangle|\psi_{x_2}\rangle\cdots|\psi_{x_n}\rangle$

We can store these  $n$  quantum states using  $2^{n[S(\rho) + \varepsilon]}$  dimensions and recover them with **average fidelity**  $\geq 1 - \delta$ .

NB. Need much less than  $2^{n[H(q) + \varepsilon]}$  (if we're to remember  $x_1 x_2 \cdots x_n$ )!!

average over the  
distribution of  
quantum states

abs(inner product)  
between given &  
recovered states

von Neumann entropy of the average state represents the space needed for compression of iid source of quantum states.

## Quantum data compression (how):

Consider the spectral decomposition of average state:

$$\rho = \sum_v p(v) |e_v\rangle\langle e_v|$$

*In its eigenbasis  $|e_v\rangle$ :*

$\rho$  is a classical random variable  $V$  with  $\text{dist}^n p$ .

$\rho^{\otimes n}$  is  $n$  iid draws of  $V$ .

Let  $T_{n,\varepsilon}$  be the typical set of  $v^n$ . (Recall what that is)

For each typical sequence  $v_1 v_2 \cdots v_n$  in  $T_{n,\varepsilon}$ , consider the state  $|e_{v_1}\rangle |e_{v_2}\rangle \cdots |e_{v_n}\rangle =: |e_{v^n}\rangle$ .

Together, they span **typical subspace  $S$**  with projector:

$$\Pi_S = \sum_{v^n \in T_{n,\varepsilon}} |e_{v^n}\rangle\langle e_{v^n}| \quad \leftarrow \text{already orthonormal}$$

$$(1) \dim S = |T_{n,\varepsilon}| \leq 2^{n(H(V)+\varepsilon)} \leq 2^{n[S(\rho)+\varepsilon]}$$

$$(2) \text{Tr}(\rho^{\otimes n} \Pi_S) = \sum_{v^n \in T_{n,\varepsilon}} p(v^n) = \text{prob}(T_{n,\varepsilon}) \geq 1-\delta.$$

expand

Encoding of quantum compression is simply projecting the state onto the typical space  $S$ .

(Reduces to classical data compression for an ensemble of orthogonal states. Why it works for an arbitrary ensemble is more delicate ...)

Suppose the  $n$  states to be compressed are

$|\psi_{x1}\rangle |\psi_{x2}\rangle \cdots |\psi_{xn}\rangle =: |\psi_{x^n}\rangle$  the normalization inserts the prob of not outputting  $|f\rangle$  automatically in the fidelity

Encoded state =  $\Pi_S |\psi_{x^n}\rangle$  (norm = prob of this happening)

Otherwise, output  $|f\rangle$  an error symbol

Average output fidelity:  $\sum_{x^n} q(x^n) \overset{\text{input}}{|\langle \psi_{x^n} |} \overset{\text{output}}{\Pi_S |\psi_{x^n}\rangle|}$

$$\geq \sum_{x^n} q(x^n) \langle \psi_{x^n} | \Pi_S | \psi_{x^n} \rangle$$

$$\overset{\substack{\text{cyclic} \\ \text{prop} \\ \text{trace}}}{=} \sum_{x^n} q(x^n) \text{Tr} [ |\psi_{x^n}\rangle \langle \psi_{x^n}| \Pi_S ]$$

$$= \text{Tr} [ \rho^{\otimes n} \Pi_S ] \geq 1 - \delta \quad \text{by previous slide}$$

More rigorous analysis using mixed state notation:

Suppose the  $n$  states to be compressed are

$$|\psi_{x1}\rangle |\psi_{x2}\rangle \cdots |\psi_{xn}\rangle =: |\psi_{xn}\rangle$$

$$\begin{aligned} \text{Encoded state} = & \Pi_S |\psi_{xn}\rangle \langle \psi_{xn}| \Pi_S \\ & + \text{Tr}[(I - \Pi_S) |\psi_{xn}\rangle \langle \psi_{xn}| (I - \Pi_S)] |f\rangle \langle f| \end{aligned}$$

Input/output fidelity:

$$\sqrt{|\langle \psi_{xn}| \Pi_S |\psi_{xn}\rangle \langle \psi_{xn}| \Pi_S |\psi_{xn}\rangle|}$$

For optimality of both quantum and classical data compression (i.e. if we use  $2^{n(S(\rho)-\eta)}$  dimensions for any  $\eta>0$ , we will not recover the states), see Nielsen and Chuang.